Signal processing on the sphere

and applications

Jason McEwen
http://www.jasonmcewen.org/

*Institute of Electrical Engineering,
Ecole Polytechnique Fédérale de Lausanne (EPFL)*

*Department of Physics and Astronomy,
University College London (UCL)*

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Outline

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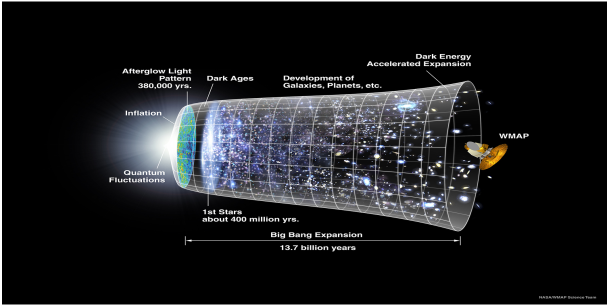
1. Cosmology
   - Big Bang
   - Cosmic microwave background
   - Observations
2. Harmonic analysis on the sphere
   - Spherical harmonic transform
   - Sampling theorems
3. Wavelets on the sphere
   - Why wavelets?
   - Continuous wavelets
   - Multiresolution analysis
4. Applications
   - Gaussianity of the CMB
   - Dark energy
   - Compression
   - Reflectance recovery

Cosmological concordance model

- Concordance model of modern cosmology emerged recently with many cosmological parameters constrained to high precision.
- General description is of a Universe undergoing accelerated expansion, containing 4% ordinary baryonic matter, 22% cold dark matter and 74% dark energy.
- Structure and evolution of the Universe constrained through cosmological observations.



Credit: WMAP Science Team

Cosmic microwave background (CMB)

- Temperature of early Universe sufficiently hot that photons had enough energy to ionise hydrogen.
- Compton scattering happened frequently $\Rightarrow$ mean free path of photons extremely small.
- Universe consisted of an opaque photon-baryon fluid.
- As Universe expanded it cooled, until majority of photons no longer had sufficient energy to ionise hydrogen.
- Photons decoupled from baryons and the Universe became essentially transparent to radiation.
- *Recombination* occurred when temperature of Universe dropped to 3000K ($\sim$400,000 years after the Big Bang).
- Photons then free to propagate largely unhindered and observed today on celestial sphere as CMB radiation.
- CMB is highly uniform over the celestial sphere, however it contains small fluctuations at a relative level of 10^{-5} due to acoustic oscillations in the early Universe.
- CMB observed on spherical manifold, hence the geometry of the sphere must be taken into account in any analysis.

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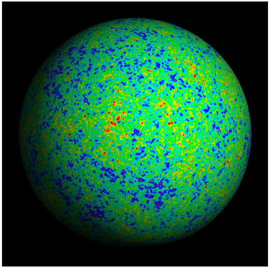

Credit: Max Tegmark

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- Provide the seeds of structure formation in our Universe.
- Cosmological concordance model explains the power spectrum of these oscillations to very high precision.
- Although a general cosmological concordance model is now established, many details remain unclear. Study of more exotic cosmological models now important.

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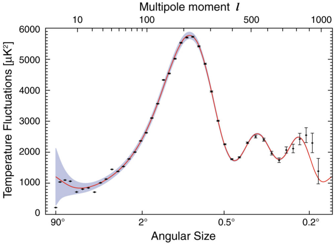

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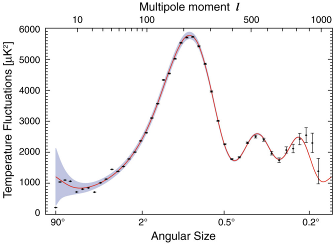

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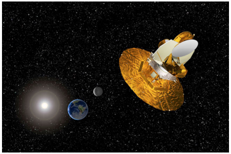
Observations of the CMB

- Full-sky observations of the CMB ongoing.

(a) COBE (launched 1989)



(b) WMAP (launched 2001)

(c) Planck (launched 2009)

Figure: Full-sky CMB observations

- Each new experiment provides dramatic improvement in precision and resolution of observations (*e.g.* COBE to WMAP illustration).

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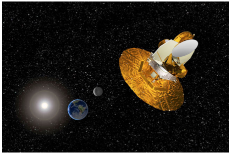

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(cobe 2 wmap movie)

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Spherical harmonics

- Consider the space of square integrable functions on the sphere $\mathrm{L}^2(\mathrm{S}^2)$, with the inner product of $f, g \in \mathrm{L}^2(\mathrm{S}^2)$ defined by

$$\langle f,\ g \rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta,\varphi)\, f(\theta,\varphi)\, g^*(\theta,\varphi) \ ,$$

where $\mathrm{d}\Omega(\theta,\varphi) = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the usual invariant measure on the sphere and (θ,φ) define spherical coordinates with colatitude $\theta \in [0,\pi]$ and longitude $\varphi \in [0, 2\pi)$. Complex conjugation is denoted by the superscript *.

- The scalar spherical harmonic functions form the canonical orthogonal basis for the space of $\mathrm{L}^2(\mathrm{S}^2)$ scalar functions on the sphere and are defined by

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\varphi} \ ,$$

for natural $\ell \in \mathbb{N}$ and integer $m \in \mathbb{Z}$, $|m| \leq \ell$, where $P_\ell^m(x)$ are the associated Legendre functions.

- Eigenfunctions of the Laplacian on the sphere: $\Delta_{\mathrm{S}^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$.

- Orthogonality relation: $\langle Y_{\ell m},\ Y_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'}$, where δ_{ij} is the Kronecker delta symbol.

- Completeness relation:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\,\delta(\varphi - \varphi') \ ,$$

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Spherical harmonic transform

- Any square integrable scalar function on the sphere $f \in \mathrm{L}^2(\mathbb{S}^2)$ may be represented by its spherical harmonic expansion:

$$\boxed{f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} \, Y_{\ell m}(\theta, \varphi) \; .}$$

- The spherical harmonic coefficients are given by the usual projection onto each basis function:

$$\boxed{f_{\ell m} = \langle f, \, Y_{\ell m} \rangle \; .}$$

- We consider signals on the sphere band-limited at L, that is signals such that $f_{\ell m} = 0, \forall \ell \geq L$
$\Rightarrow$ summations may be truncated at $L - 1$.
- For a band-limited signal, can we compute $f_{\ell m}$ exactly?
$\rightarrow$ Sampling theorems on the sphere!
- Aside: Generalise to spin functions on the sphere.
Square integrable spin functions on the sphere ${}_sf \in \mathrm{L}^2(\mathbb{S}^2)$, with integer spin $s \in \mathbb{Z}$, are defined by their behaviour under local rotations. By definition, a spin function transforms as

$${}_sf'(\theta, \varphi) = \mathrm{e}^{-\mathrm{i}s\chi} \, {}_sf(\theta, \varphi)$$

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Sampling theorems on the sphere: state-of-the-art

- In-exact spherical harmonic transforms exist for a variety of pixelisations of the sphere.
  - `HEALpix` (Gorski *et al.* 2005)
  - `IGLOO` (Crittenden & Turok 1998)

$\rightarrow$ Do NOT lead to sampling theorems on the sphere!

- Driscoll & Healy (1994) sampling theorem
  - Equiangular pixelisation of the sphere
  - Require $\sim 4L^2$ samples on the sphere
  - Semi-naive algorithm with complexity $\mathcal{O}(L^3)$
    (algorithms with lower scaling exist but they are not generally stable)
  - Require a precomputation or otherwise restricted use of Wigner recursions

- Gauss-Legendre sampling theorem
  - Not generally so well-know (no published work)
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Sampling theorems on the sphere: a novel sampling theorem

- We have developed a new sampling theorem and corresponding fast algorithms by performing a factoring of rotations and then by associating the sphere with the torus through a periodic extension.
- Similar (in flavour but not detail!) to making a periodic extension in θ of a function f on the sphere.

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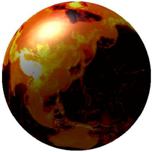
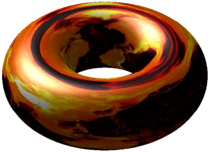
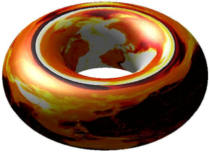


(a) Function on sphere



(b) Even function on torus



(c) Odd function on torus

Figure: Associating functions on the sphere and torus

Sampling theorems on the sphere: a novel sampling theorem

- By a factoring of rotations (Wigner decomposition), a reordering of summations and a separation of variables, the inverse transform of ${}_s f$ may be written:

Inverse spherical harmonic transform

$$ {}_s f(\theta, \varphi) = \sum_{m=-(L-1)}^{L-1} {}_s F_m(\theta)\, e^{im\varphi} $$

$$ {}_s F_m(\theta) = \sum_{m'=-(L-1)}^{L-1} {}_s F_{mm'}\, e^{im'\theta} $$

$$ {}_s F_{mm'} = (-1)^s\, i^{-(m+s)} \sum_{\ell=0}^{L-1} \sqrt{\frac{2\ell+1}{4\pi}}\, \Delta^{\ell}_{m'm}\, \Delta^{\ell}_{m',-s}\, {}_s f_{\ell m} $$

where $\Delta^{\ell}_{mn} \equiv d^{\ell}_{mn}(\pi/2)$ are the reduced Wigner functions evaluated at $\pi/2$.

Sampling theorems on the sphere: a novel sampling theorem

- By a factoring of rotations (Wigner decomposition), a reordering of summations and a separation of variables, the forward transform of ${}_sf$ may be written:

Forward spherical harmonic transform

$$ {}_sf_{\ell m} = (-1)^s \, \mathrm{i}^{m+s} \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'=-(L-1)}^{L-1} \Delta^{\ell}_{m' m} \, \Delta^{\ell}_{m', -s} \, {}_sG_{mm'} $$

$$ {}_sG_{mm'} = \int_0^{\pi} \mathrm{d}\theta \sin\theta \, {}_sG_m(\theta) \, \mathrm{e}^{-\mathrm{i}m'\theta} $$

$$ {}_sG_m(\theta) = \int_0^{2\pi} \mathrm{d}\varphi \, {}_sf(\theta, \varphi) \, \mathrm{e}^{-\mathrm{i}m\varphi} $$

- Recasting the forward and inverse spherical harmonic transforms in this manner is no more efficient or accurate than the original formulation.
- However, it highlights similarities with Fourier series representations and reduces the problem of finding an exact quadrature rule to the calculation of ${}_sG_{mm'}$ only.
- The Fourier series expansion is only defined for periodic functions; thus, to recast these expressions in a form amenable to the application of Fourier transforms we must make a periodic extension in colatitude θ.

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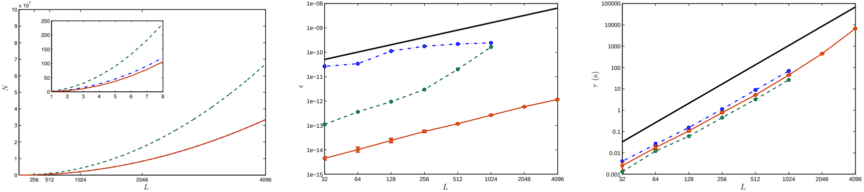
Sampling theorems on the sphere: a novel sampling theorem

- Properties of our new sampling theorem
  - Equiangular pixelisation of the sphere
  - Require $\sim 2L^2$ samples on the sphere (and still fewer than Gauss-Legendre sampling)
  - Exploit fast Fourier transforms to yield a fast algorithm with complexity $\mathcal{O}(L^3)$
  - No precomputation and very flexible regarding use of Wigner recursions
  - Extends to spin function on the sphere with no change in complexity or computation time



(a) Number of samples (b) Numerical accuracy (c) Computation time

Figure: Performance of our sampling theorem (MW=red; DH=green; Gauss-Legendre=blue)

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Morlet and Grossman (1981)

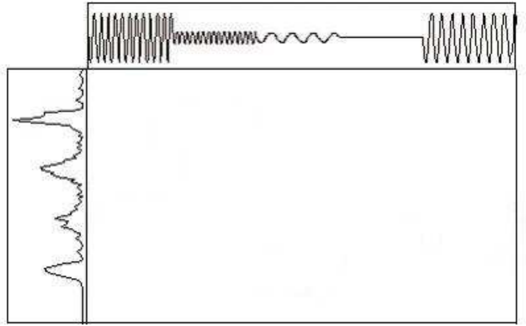

Figure: Fourier vs wavelet transform (image from `http://www.wavelet.org/tutorial/`)

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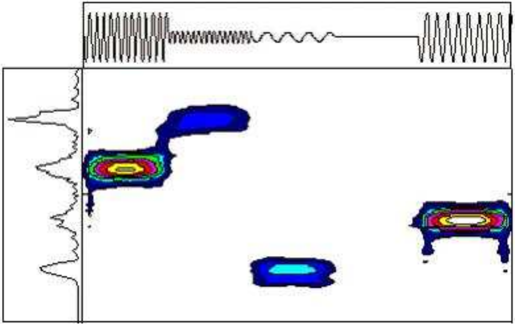

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Recall wavelet transform in Euclidean space

- Project signal onto wavelets

$$\mathcal{W}^f(a,b) = \langle f, \psi_{a,b} \rangle = |a|^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}t \, f(t) \, \psi^* \left(\frac{t-b}{a} \right),$$

  where $\psi_{a,b} = |a|^{-1/2} \psi(\frac{t-b}{a})$.

- Synthesis signal from wavelet coefficients

$$f(t) = C_\psi^{-1} \int_{-\infty}^{\infty} \mathrm{d}b \int_0^{\infty} \frac{\mathrm{d}a}{a^2} \, \mathcal{W}^f(a,b) \psi_{a,b}(t).$$

- Admissibility condition to ensure perfect reconstruction

$$0 < C_\psi \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{|k|} \, |\hat{\psi}(k)|^2 < \infty.$$

- Construct on sphere in analogous manner.

Wavelets on the sphere

- Follow construction derived by Antoine and Vandergheynst (1998) (reintroduced by Wiaux (2005)).
- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.
- The natural extension of translations to the sphere are rotations. Characterised by the elements of the rotation group $\mathrm{SO}(3)$, which parameterise in terms of the three Euler angles $\rho = (\alpha, \beta, \gamma)$. Rotation of a function f on the sphere is defined by

$$[\mathcal{R}(\rho)f](\omega) = f(\rho^{-1}\omega), \quad \rho \in \mathrm{SO}(3) .$$

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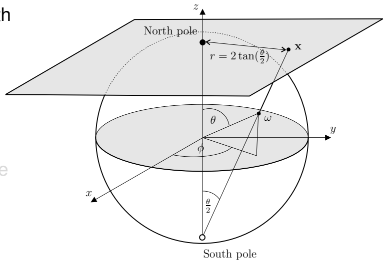
Stereographic projection

- Apply stereographic projection to build an association with the plane.



- Stereographic projection operator is defined by $\Pi : \omega \to \boldsymbol{x} = \Pi\omega = (r(\theta), \varphi)$ where $r = 2\tan(\theta/2)$, $\omega \equiv (\theta, \varphi) \in \mathrm{S}^2$ and $\boldsymbol{x} \in \mathbb{R}^2$ is a point in the plane, denoted here by the polar coordinates (r, φ). The inverse operator is $\Pi^{-1} : \boldsymbol{x} \to \omega = \Pi^{-1}\boldsymbol{x} = (\theta(r), \varphi)$, where $\theta(r) = 2\tan^{-1}(r/2)$.

- Define the action of the stereographic projection operator on functions on the plane and sphere. Consider the space of square integrable functions in $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}^2\boldsymbol{x})$ on the plane and $\mathrm{L}^2(\mathrm{S}^2, \mathrm{d}\Omega(\omega))$ on the sphere.
    - The action of the stereographic projection operator $\Pi : f \in \mathrm{L}^2(\mathrm{S}^2, \mathrm{d}\Omega(\omega)) \to p = \Pi f \in \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}^2\boldsymbol{x})$ on functions is defined as
    $$p(r, \varphi) = (\Pi f)(r, \varphi) = (1 + r^2/4)^{-1} f(\theta(r), \varphi) .$$
    - The inverse stereographic projection operator $\Pi^{-1} : p \in \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}^2\boldsymbol{x}) \to f = \Pi^{-1}p \in \mathrm{L}^2(\mathrm{S}^2, \mathrm{d}\Omega(\omega))$ on functions is then
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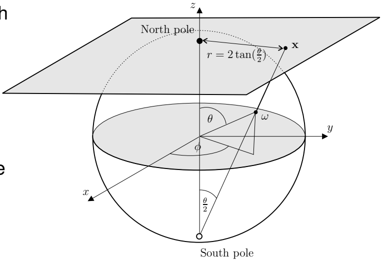

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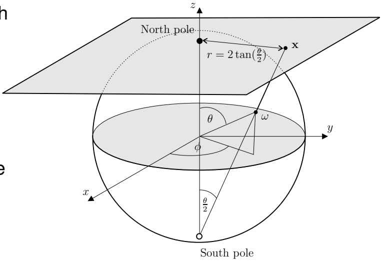

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Dilation on the sphere

- The spherical dilation operator $\mathcal{D}(a) : f(\omega) \rightarrow [\mathcal{D}(a)f](\omega)$ in $\mathrm{L}^2(\mathrm{S}^2, \,\mathrm{d}\Omega(\omega))$ is defined as the conjugation by Π of the Euclidean dilation $d(a)$ in $\mathrm{L}^2(\mathbb{R}^2, \,\mathrm{d}^2\boldsymbol{x})$ on tangent plane at north pole:

$$\mathcal{D}(a) \equiv \Pi^{-1} \, d(a) \, \Pi \, .$$

- Spherical dilation given by

$$[\mathcal{D}(a)f](\omega) = [\lambda(a, \theta, \varphi)]^{1/2} \, f(\omega_{1/a}) \, ,$$

where $\omega_a = (\theta_a, \varphi)$ and $\tan(\theta_a/2) = a \tan(\theta/2)$.

- Cocycle of a spherical dilation is defined by

$$\lambda(a, \theta, \varphi) \equiv \frac{4a^2}{[(a^2 - 1)\cos\theta + (a^2 + 1)]^2} \, .$$

Wavelet analysis formula

- Wavelets on the sphere may now be constructed from rotations and dilations of a mother spherical wavelet $\Phi \in \mathrm{L}^2(\mathrm{S}^2, \,\mathrm{d}\Omega(\omega))$. The corresponding wavelet family $\{\Phi_{a,\rho} \equiv \mathcal{R}(\rho)\mathcal{D}(a)\Phi : \rho \in \mathrm{SO}(3), \, a \in \mathbb{R}^+_*\}$ provides an over-complete set of functions in $\mathrm{L}^2(\mathrm{S}^2, \,\mathrm{d}\Omega(\omega))$.
- The CSWT of $f \in \mathrm{L}^2(\mathrm{S}^2, \,\mathrm{d}\Omega(\omega))$ is given by the projection on to each wavelet atom in the usual manner:

$$\boxed{\widehat{\mathcal{W}}^f_\Phi(a, \rho) = \langle f, \Phi_{a,\rho} \rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\omega)\, f(\omega)\, \Phi^*_{a,\rho}(\omega) \,,}$$

  where $\mathrm{d}\Omega(\omega) = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi$ is the usual invariant measure on the sphere.
- Transform general in the sense that all orientations in the rotation group $\mathrm{SO}(3)$ are considered, thus directional structure is naturally incorporated.
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- The synthesis of a signal on the sphere from its wavelet coefficients is given by

$$\boxed{f(\omega) = \int_0^\infty \frac{\mathrm{d}a}{a^3} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) \widehat{\mathcal{W}}_\Phi^f(a, \rho) \; [\mathcal{R}(\rho) \widehat{L}_\Phi \Phi_a](\omega) \; ,}$$

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$$(\widehat{L}_\Phi g)_{\ell m} \equiv g_{\ell m} / \widehat{C}_\Phi^\ell$$

  on the spherical harmonic coefficients of functions $g \in \mathrm{L}^2(\mathrm{S}^2, \, \mathrm{d}\Omega(\omega))$.

- In order to ensure the perfect reconstruction of a signal synthesised from its wavelet coefficients, the admissibility condition

$$\boxed{0 < \widehat{C}_\Phi^\ell \equiv \frac{8\pi^2}{2\ell+1} \sum_{m=-\ell}^{\ell} \int_0^\infty \frac{\mathrm{d}a}{a^3} \mid (\Phi_a)_{\ell m} \mid^2 < \infty}$$

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- Correspondence principle between spherical and Euclidean wavelets states that the inverse stereographic projection of an *admissible* wavelet on the plane yields an *admissible* wavelet on the sphere (proved by Wiaux *et al.* 2005)

- Mother wavelets on sphere constructed from the projection of mother Euclidean wavelets defined on the plane:

$$\Phi = \Pi^{-1} \Phi_{\mathbb{R}^2} \ ,$$

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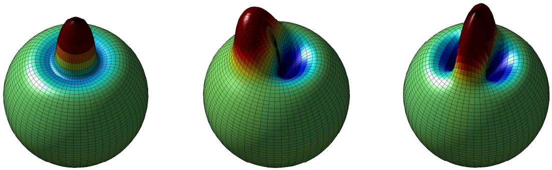

(a) SMHW (b) SBW (c) S2GDW

Figure: Spherical wavelets at scale $a, b = 0.2$.

Multiresolution analysis on the sphere

- Define multiresolution analysis on the sphere in an analogous manner to Euclidean framework.
- Define approximation spaces on the sphere $V_j \subset \mathrm{L}^2(\mathrm{S}^2)$
- Construct the nested hierarchy of approximation spaces

$$V_1 \subset V_2 \subset \cdots \subset V_J \subset \mathrm{L}^2(\mathrm{S}^2) \, ,$$

where coarser (finer) approximation spaces correspond to a lower (higher) resolution level j.
- For each space V_j we define a basis with basis elements given by the *scaling functions* $\varphi_{j,k} \in V_j$, where the k index corresponds to a translation on the sphere.
- Define detail space W_j to be the orthogonal complement of V_j in V_{j+1}, *i.e.* $V_{j+1} = V_j \oplus W_j$.
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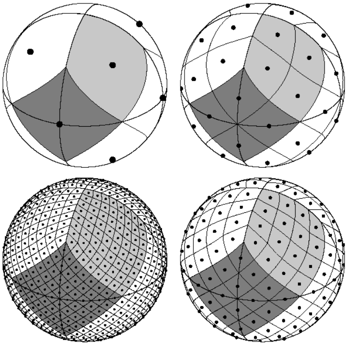
Hierarchical pixelisation of the sphere

- Relate generic multiresolution decomposition to HEALPix hierarchical pixelisation of the sphere.



Credit: Krzysztof Gorski

Haar wavelets on the sphere

- Let V_j correspond to a `HEALPix` pixelised sphere with resolution parameter $N_{\text{side}} = 2^{j-1}$.
- Define the scaling function $\varphi_{j,k}$ at level j to be constant for pixel k and zero elsewhere:

$$\varphi_{j,k}(\omega) \equiv \begin{cases} 1/\sqrt{A_j} & \omega \in P_{j,k} \\ 0 & \text{elsewhere} . \end{cases}$$

- Orthonormal basis for the wavelet space W_j given by the following wavelets:

$$\psi^0_{j,k}(\omega) \equiv \left[\varphi_{j+1,k_0}(\omega) - \varphi_{j+1,k_1}(\omega) + \varphi_{j+1,k_2}(\omega) - \varphi_{j+1,k_3}(\omega)\right]/2 ;$$
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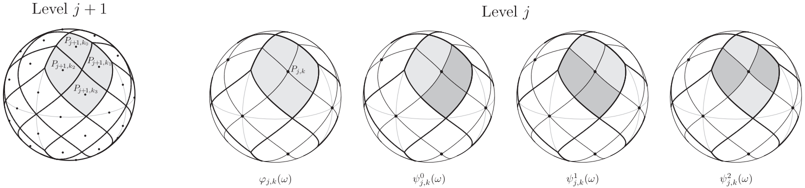

Figure: Haar scaling function $\varphi_{j,k}(\omega)$ and wavelets $\psi^m_{j,k}(\omega)$

Haar wavelets on the sphere

- Multiresolution decomposition of a function defined on a HEALPix data-sphere at resolution J, *i.e.* $f_J \in V_J$ proceeds as follows.
- Approximation coefficients at the coarser level j are given by the projection of f_{j+1} onto the scaling functions $\varphi_{j,k}$:

$$\lambda_{j,k} = \int_{S^2} f_{j+1}(\omega)\ \varphi_{j,k}(\omega)\ \mathrm{d}\Omega\ .$$

- Detail coefficients at level j are given by the projection of f_{j+1} onto the wavelets $\psi^m_{j,k}$:

$$\gamma^m_{j,k} = \int_{S^2} f_{j+1}(\omega)\ \psi^m_{j,k}(\omega)\ \mathrm{d}\Omega\ .$$

- The function $f_J \in V_J$ may then be synthesised from its approximation and detail coefficients:

$$f_J(\omega) = \sum_{k=0}^{N_{J_0}-1} \lambda_{J_0 k}\varphi_{J_0 k}(\omega) + \sum_{j=J_0}^{J-1}\sum_{k=0}^{N_j-1}\sum_{m=0}^{2}\gamma^m_{j,k}\psi^m_{j,k}(\omega)\ .$$

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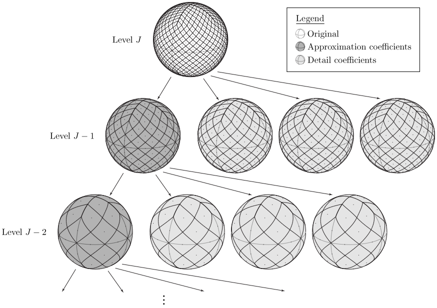


Figure: Haar multiresolution decomposition

- The function $f_J \in V_J$ may then be synthesised from its approximation and detail coefficients:

$$f_J(\omega) = \sum_{k=0}^{N_{J_0}-1} \lambda_{J_0 k}\varphi_{J_0 k}(\omega) + \sum_{j=J_0}^{J-1}\sum_{k=0}^{N_j-1}\sum_{m=0}^{2} \gamma^m_{j,k}\psi^m_{j,k}(\omega)\ .$$

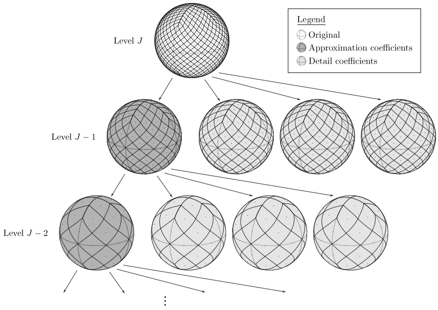
Haar wavelets on the sphere

- Multiresolution decomposition of a function defined on a HEALPix data-sphere at resolution J, *i.e.* $f_J \in V_J$ proceeds as follows.
- Approximation coefficients at the coarser level j are given by the projection of f_{j+1} onto the scaling functions $\varphi_{j,k}$:

$$\lambda_{j,k} = \int_{S^2} f_{j+1}(\omega)\ \varphi_{j,k}(\omega)\ \mathrm{d}\Omega\ .$$

- Detail coefficients at level j are given by the projection of f_{j+1} onto the wavelets $\psi^m_{j,k}$:

$$\gamma^m_{j,k} = \int_{S^2} f_{j+1}(\omega)\ \psi^m_{j,k}(\omega)\ \mathrm{d}\Omega\ .$$



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Outline

Gaussianity of the CMB

- Statistics of primordial fluctuations provide a useful mechanism for distinguishing between various scenarios of the early Universe, such as various models of inflation.
- Primordial fluctuations give rise to the CMB anisotropies.
- In the simplest inflationary scenarios, primordial perturbations seed Gaussian temperature fluctuations in the CMB.
- However, this is not the case for non-standard inflationary models.
- Evidence of non-Gaussianity in the CMB anisotropies would therefore have profound implications for the standard cosmological concordance model.
- Probe WMAP observations of the CMB for evidence of non-Gaussianity.

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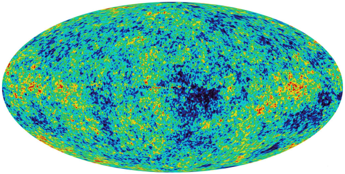

Wavelet analysis of Gaussianity of the CMB

- Various physical processes manifest at different scales and locations, hence employ wavelet analysis to probe CMB.
- Wavelet coefficients of Gaussian signal remain Gaussian distributed (due to linearity of wavelet transform).
- Examine the skewness and kurtosis of wavelet coefficients.
- Compare to Monte Carlo simulations of Gaussian CMB realisations.
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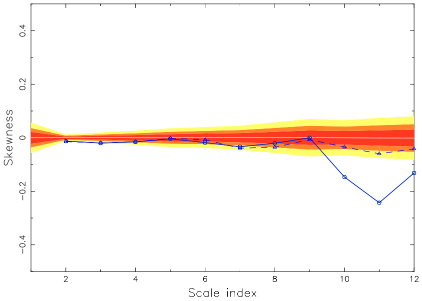

Figure: Skewness of wavelet coefficients

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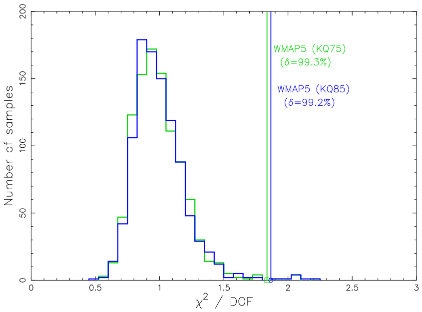
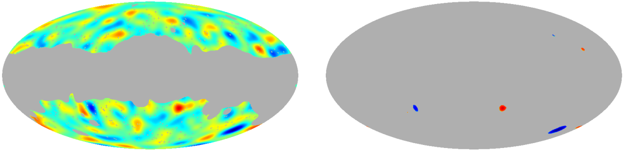


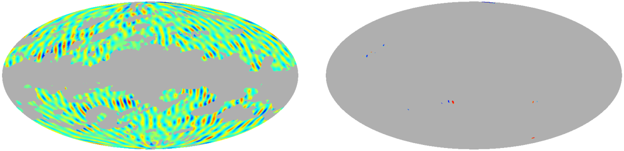
Figure: χ^2 of skewness of wavelet coefficients

Localisation of non-Gaussian features in the CMB

- Localise regions that contribute most significantly to the non-Gaussian signal.
- Detection of the “cold spot” anomaly in the CMB.
- Various new cosmology models constructed in attempt to explain the cold spot.



(a) SMHW coefficients



(b) SMW coefficients

Figure: Spherical wavelet coefficient maps (left) and thresholded maps (right)

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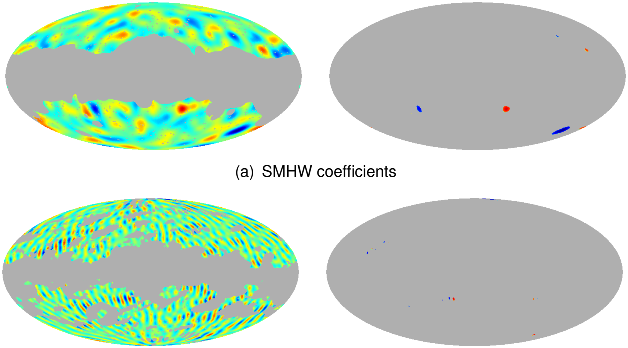

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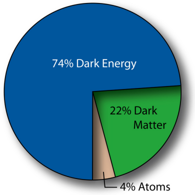
Dark energy

- Universe consists of ordinary baryonic matter, cold dark matter and dark energy.
- Dark energy represents energy density of empty space. Modelled by a cosmological fluid with negative pressure acting as a repulsive force.
- Evidence for dark energy provided by observations of CMB, supernovae and large scale structure of Universe.



Credit: WMAP Science Team

- However, a consistent model in the framework of particle physics lacking. Indeed, attempts to predict a cosmological constant obtain a value that is too large by a factor of $\sim 10^{120}$.
- Dark energy dominates our Universe but yet we know very little about its nature and origin.
- Verification of dark energy by independent physical methods of considerable interest.
- Independent methods may also prove more sensitive probes of properties of dark energy.

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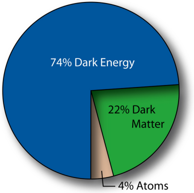

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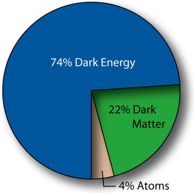

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Integrated Sachs-Wolfe (ISW) effect

(ball sim constant movie) (ball sim evolving movie)

Figure: ISW effect analogy

- CMB photons blue (red) shifted when fall into (out of) potential wells.
- Evolution of potential during photon propagation $\rightarrow$ net change in photon energy.
- Gravitation potentials constant w.r.t. conformal time in matter dominated universe.
- Deviation from matter domination due to curvature or dark energy causes potentials to evolve with time $\rightarrow$ secondary anisotropy induced in CMB.

Detecting the ISW effect

- WMAP shown universe is (nearly) flat.
- Detection of ISW effect $\Rightarrow$ direct evidence for dark energy.
- Cannot isolate the ISW signal from CMB anisotropies easily.
- Instead, detect by cross-correlating CMB anisotropies with tracers of large scale structure.
  (Crittenden & Turok 1996
- Wavelets ideal analysis tool to search for correlation induced by ISW effect since signal manifest at different scales and locations.
  (Pioneered by Vielva *et al.* 2005, followed by JDM *et al.* 2006, JDM *et al.* 2007 and others.)
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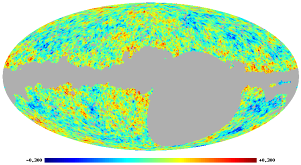
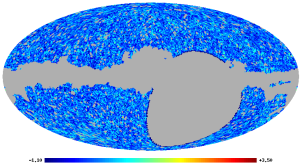


(a) WMAP



(b) NVSS

Figure: WMAP and NVSS maps after application of the joint mask

Detection of the ISW effect with wavelets

- Significant correlation detected between the WMAP and NVSS data.
- Foreground contamination and instrumental systematics ruled out as source of the correlation $\Rightarrow$ correlation due to ISW effect.
- Direct observational evidence for dark energy.

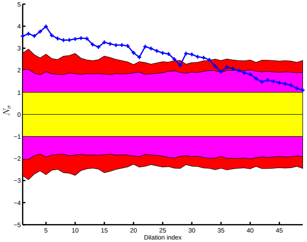


Figure: Wavelet correlation

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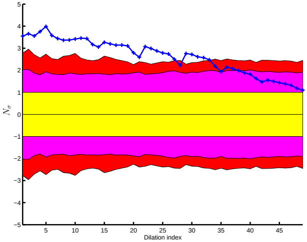

Figure: Wavelet correlation

Constraining dark energy with wavelets

- Possible to use positive detection of the ISW effect to constrain parameters of cosmological models that describe dark energy:
  - Proportional energy density Ω_Λ.
  - Equation of state parameter w relating pressure and density of cosmological fluid that models dark energy, *i.e.* $p = w\rho$.
- Parameter estimates of $\Omega_\Lambda = 0.63^{+0.18}_{-0.17}$ and $w = -0.77^{+0.35}_{-0.36}$ computed from the mean of the marginalised distributions (consistent with other analysis techniques and data sets).

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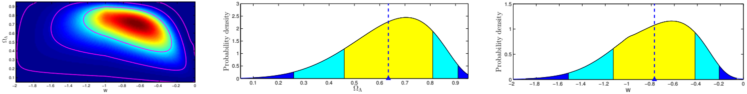
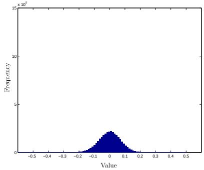


(a) Full likelihood surface (b) Marginalised distribution for Ω_Λ (c) Marginalised distribution for w

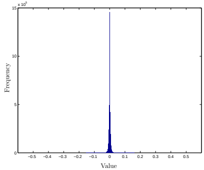
Figure: Dark energy likelihoods

Compression of data on the sphere

- Current and forthcoming observations of the CMB of considerable size.
- Haar wavelet transform to compress energy content.



(a) Original data



(b) Wavelet coefficients

Figure: Histograms

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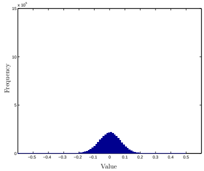
Lossless compression algorithm

1. Haar wavelet transform on sphere
2. Quantise detail coefficients to numerical precision (precision parameter p)
3. Huffman encoding



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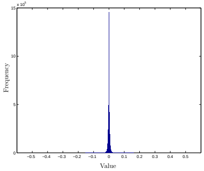

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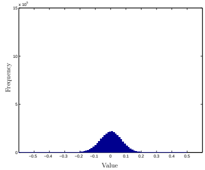
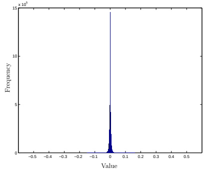
Lossy compression algorithm

1. Haar wavelet transform on sphere
2. Thresholding
3. Quantise detail coefficients to numerical precision
4. Run-length encoding (RLE)
5. Huffman encoding



(a) Original data

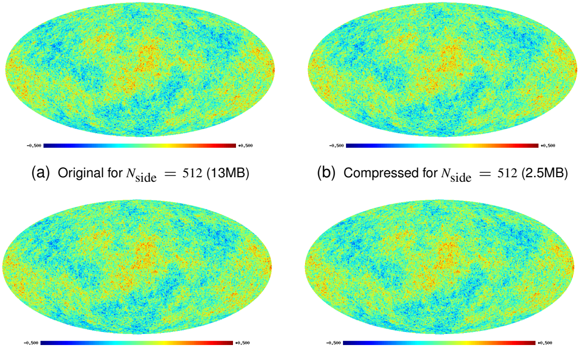


(b) Wavelet coefficients

Figure: Histograms

Compression of CMB data

- Lossless to a user specified numerical precision only.



(a) Original for $N_{\rm side} = 512$ (13MB)

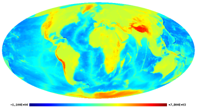
(b) Compressed for $N_{\rm side} = 512$ (2.5MB)

(c) Original for $N_{\rm side} = 1024$ (50MB)

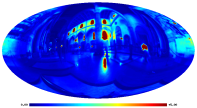
(d) Compressed for $N_{\rm side} = 1024$ (9.1MB)

Figure: Lossless compression of simulated Gaussian CMB data

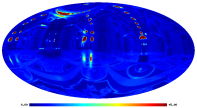
Lossy compression applications



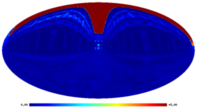
(a) Earth: original (13MB)



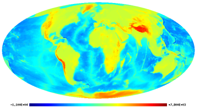
(b) Galileo: original (3.2MB)



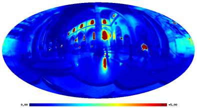
(c) St Peter's: original (3.2MB)



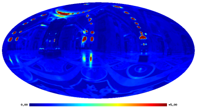
(d) Uffizi: original (3.2MB)



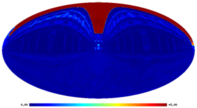
(e) Earth: lossless (1.4MB)



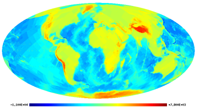
(f) Galileo: lossless (0.21MB)



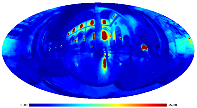
(g) St Peter's: lossless (0.20MB)



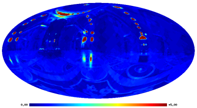
(h) Uffizi: lossless (0.19MB)



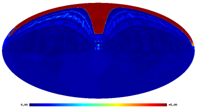
(i) Earth: lossy (0.33MB)



(j) Galileo: lossy (0.07MB)



(k) St Peter's: lossy (0.08MB)



(l) Uffizi: lossy (0.10MB)

Figure: Compressed data for lossy compression applications

Reflectance recovery

- Functions encountered in computer graphics are typically defined over directions $\rightarrow$ data on the sphere.
- Illumination maps inherently defined on the sphere (*i.e.* over directions):

  Let $L(\omega)$ denote the illumination function, where $\omega = (\theta, \varphi) \in S^2$.

- Bidirectional reflectance distributions functions (BRDFs) inherently defined on the product of spheres:

  Let $\tilde{\Gamma}(\omega_{\rm i}, \omega_{\rm o})$ denote the BRDF, where $\omega_{\rm i}$ and $\omega_{\rm o}$ are incoming and outgoing directions respectively.

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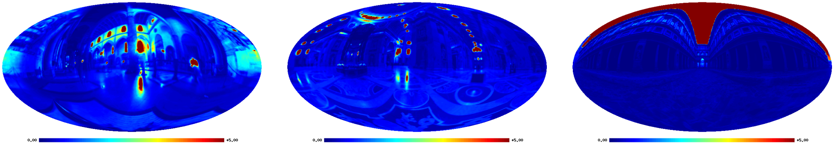

(a) Galileo's Tomb (b) St Peter's Basilica (c) Uffizi Gallery

Figure: Illumination maps [http://www.debevec.org/Probes]

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Theoretical framework for reflection

- Adopt the framework of Ramamoorthi & Hanrahan (2001).
- Outgoing radiance given by

$$B(\rho, \omega_{\mathrm{o}}) = \int_{S^2} L(\omega_{\mathrm{i}}') \, \tilde{\Gamma}(\omega_{\mathrm{i}}, \omega_{\mathrm{o}}) \cos(\theta_{\mathrm{i}}) \; \mathrm{d}\Omega(\omega_{\mathrm{i}}) \; ,$$

where the prime denotes global coordinates, $\rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)$ is the orientation of the surface element (*cf.* surface normal) and $\mathrm{d}\Omega(\omega) = \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the usual rotation invariant measure on the sphere.

- Local coordinates and global coordinates related through rotation $\mathcal{R}$ about surface element $\omega' = \mathcal{R}(\rho)\, \omega$, hence in a consistent coordinate frame we obtain

$$B(\rho, \omega_{\mathrm{o}}) = \int_{S^2} (\mathcal{R}^{-1}(\rho) L)(\omega_{\mathrm{i}}) \; \Gamma(\omega_{\mathrm{i}}, \omega_{\mathrm{o}}) \; \mathrm{d}\Omega(\omega_{\mathrm{i}}) \; .$$

- Through the rotation invariance of the measure on the sphere we obtain the reflection equation

$$\boxed{B(\rho, \omega_{\mathrm{o}}) = \int_{S^2} L(\omega_{\mathrm{i}}) \; (\mathcal{R}(\rho)\Gamma)(\omega_{\mathrm{i}}, \omega_{\mathrm{o}}) \; \mathrm{d}\Omega(\omega_{\mathrm{i}})} \; .$$

- Assumptions: homogenous object; known geometry; distant illumination; no inter-reflection.

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$$B(\rho, \omega_{\rm o}) = \int_{S^2} L(\omega_{\rm i}') \, \tilde{\Gamma}(\omega_{\rm i}, \omega_{\rm o}) \cos(\theta_{\rm i}) \, \mathrm{d}\Omega(\omega_{\rm i}) \, ,$$

  where the prime denotes global coordinates, $\rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)$ is the orientation of the surface element (*cf.* surface normal) and $\mathrm{d}\Omega(\omega) = \sin(\theta) \, \mathrm{d}\theta \, \mathrm{d}\varphi$ is the usual rotation invariant measure on the sphere.
- Local coordinates and global coordinates related through rotation $\mathcal{R}$ about surface element $\omega' = \mathcal{R}(\rho) \, \omega$, hence in a consistent coordinate frame we obtain

$$B(\rho, \omega_{\rm o}) = \int_{S^2} (\mathcal{R}^{-1}(\rho) L)(\omega_{\rm i}) \, \Gamma(\omega_{\rm i}, \omega_{\rm o}) \, \mathrm{d}\Omega(\omega_{\rm i}) \, .$$

- Through the rotation invariance of the measure on the sphere we obtain the reflection equation

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- Assumptions: homogenous object; known geometry; distant illumination; no inter-reflection.

Theoretical framework for reflection

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Precomputed radiance transfer (PRT) using wavelets on the sphere

- Consider the forward rending problem.
- Adopt the framework of Ng *et al.* (2004).

  Compute the outgoing radiance for each vertex of the object $\boldsymbol{x}$ and incorporate a visibility function $V(\boldsymbol{x}, \omega_{\mathrm{i}})$. In this setting the reflection equation becomes

$$B(\boldsymbol{x}, \omega_{\mathrm{o}}) = \int_{\mathrm{S}^2} L(\omega_{\mathrm{i}}) \, \big(\mathcal{R}(\rho(\boldsymbol{x}))\Gamma\big)(\omega_{\mathrm{i}}, \omega_{\mathrm{o}}) \, V(\boldsymbol{x}, \omega_{\mathrm{i}}) \, \mathrm{d}\Omega(\omega_{\mathrm{i}}) \, .$$

- Very high computational complexity $\rightarrow$ infeasible for practical purposes.
- Ng *et al.* resolve this issue by using planar wavelets.
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  Take geometry of the sphere into account $\rightarrow$ greater accuracy and efficiency.

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Reflectance recovery: current approaches

- Consider the inverse rendering problem, *i.e.* recover the BRDF from known outgoing radiance and known background illumination.
- Reflection equation again:

$$\boxed{B(\rho, \omega_\mathrm{o}) = \int_{\mathrm{S}^2} L(\omega_\mathrm{i}) \, (\mathcal{R}(\rho)\Gamma)(\omega_\mathrm{i}, \omega_\mathrm{o}) \, \mathrm{d}\Omega(\omega_\mathrm{i})}$$

- Many reflectance acquisition systems involve specialised lighting configurations which considerably simplify the inverse problem.

  For example, point light source: $L(\omega_\mathrm{i}) = \delta(\omega_\mathrm{i} - \bar{\omega}) \Rightarrow B(\rho, \omega_\mathrm{o}) = (\mathcal{R}(\rho)\Gamma)(\bar{\omega}, \omega_\mathrm{o})$.
- Disadvantages of current acquisition systems:
  - Slow
  - Specialised apparatus
  - Carefully controlled environment

  $\rightarrow$ Painstaking and expensive process

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- Solve the reflection equation directly, under natural environmental illumination.
- Difficult inverse problem.
- Shift the complexity from elaborate experiment apparatus and procedures to more complicated algorithms but much simpler acquisition.
- Considered previously by Ramamoorthi & Hanrahan but limited to very low frequencies due to computational complexity $\rightarrow$ focused on theoretical implications.
- Fast, spherical wavelet based methods.
  $\rightarrow$ much higher frequencies computationally tractable
  $\rightarrow$ greater accuracy and performance
- Proposed acquisition system.
- Advantages of proposed acquisition system:
  - Fast acquisition
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