Sampling theorems and compressive sensing on the sphere

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26 January 2012 :: BASP seminar :: EPFL, Switzerland

Outline

Spherical harmonics

- Consider the space of square integrable functions on the sphere $\mathrm{L}^2(\mathrm{S}^2)$, with the inner product of $f, g \in \mathrm{L}^2(\mathrm{S}^2)$ defined by

$$\langle f,\ g\rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta,\varphi)\, f(\theta,\varphi)\, g^*(\theta,\varphi)\ ,$$

  where $\mathrm{d}\Omega(\theta,\varphi) = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the usual invariant measure on the sphere and (θ,φ) define spherical coordinates with colatitude $\theta \in [0,\pi]$ and longitude $\varphi \in [0, 2\pi)$. Complex conjugation is denoted by the superscript *.

- The scalar spherical harmonic functions form the canonical orthogonal basis for the space of $\mathrm{L}^2(\mathrm{S}^2)$ scalar functions on the sphere and are defined by

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, \mathrm{e}^{im\varphi}\ ,$$

  for natural $\ell \in \mathbb{N}$ and integer $m \in \mathbb{Z}$, $|m| \leq \ell$, where $P_\ell^m(x)$ are the associated Legendre functions.

- Eigenfunctions of the Laplacian on the sphere: $\Delta_{\mathrm{S}^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$.

- Orthogonality relation: $\langle Y_{\ell m},\ Y_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}$, where δ_{ij} is the Kronecker delta symbol.

- Completeness relation:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y_{\ell m}^*(\theta',\varphi') = \delta(\cos\theta - \cos\theta')\, \delta(\varphi-\varphi')\ ,$$

  where $\delta(x)$ is the Dirac delta function.

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Spherical harmonic transform

- Any square integrable scalar function on the sphere $f \in \mathrm{L}^2(\mathrm{S}^2)$ may be represented by its spherical harmonic expansion:

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}(\theta, \varphi) \,.$$

- The spherical harmonic coefficients are given by the usual projection onto each basis function:

$$f_{\ell m} = \langle f,\, Y_{\ell m} \rangle = \int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta, \varphi)\, f(\theta, \varphi)\, Y_{\ell m}^{*}(\theta, \varphi) \,.$$

- We consider signals on the sphere band-limited at L, that is signals such that $f_{\ell m} = 0,\ \forall \ell \geq L$
$\Rightarrow$ summations may be truncated to $L - 1$.

- Aside: Generalise to spin functions on the sphere.
Square integrable spin functions on the sphere ${}_s f \in \mathrm{L}^2(\mathrm{S}^2)$, with integer spin $s \in \mathbb{Z}$, $|s| \leq \ell$, are defined by their behaviour under local rotations. By definition, a spin function transforms as

$${}_s f'(\theta, \varphi) = \mathrm{e}^{-\mathrm{i}s\chi}\ {}_s f(\theta, \varphi)$$

under a local rotation by χ, where the prime denotes the rotated function.

- Exact in the continuous setting but require sampling theorems on the sphere for discrete signals.

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  Square integrable spin functions on the sphere ${}_sf \in \mathrm{L}^2(\mathrm{S}^2)$, with integer spin $s \in \mathbb{Z}$, $|s| \leq \ell$, are defined by their behaviour under local rotations. By definition, a spin function transforms as

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Driscoll & Healy sampling theorem (DH)

- The DH sampling theorem gives an explicit quadrature rule for the spherical harmonic transform:

$$f_{\ell m} = \sum_{t=0}^{2L-1} \sum_{p=0}^{2L-1} q_{\rm DH}(\theta_t)\, f(\theta_t, \varphi_p)\, Y_{\ell m}^*(\theta_t, \varphi_p) \,,$$

  where the sample positions are defined by $\theta_t = \pi t / 2L$, for $t = 0, \ldots, 2L-1$, and $\varphi_p = \pi p / L$, for $p = 0, \ldots, 2L-1$

  $\Rightarrow \quad N_{\rm DH} = (2L-1)2L + 1 \sim 4L^2$ samples on the sphere.

- The quadrature weights are defined implicitly by the solution to

$$\sum_{t=0}^{2L-1} q_{\rm DH}(\theta_t)\, P_\ell(\cos\theta_t) = \frac{2\pi}{L}\, \delta_{\ell 0} \,, \quad \forall \ell < 2L \,,$$

  and are given explicitly by

$$q_{\rm DH}(\theta_t) = \frac{2\pi}{L^2} \sin\theta_t \sum_{k=0}^{L-1} \frac{\sin((2k+1)\theta_t)}{2k+1} \,.$$

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- The exactness of the quadrature rule is proved by considering the sampling distribution of Dirac delta functions defined by

$$s(\theta, \varphi) = \sum_{t=0}^{2L-1} \sum_{p=0}^{2L-1} q_{\rm DH}(\theta_t)\, \delta(\cos\theta - \cos\theta_t)\, \delta(\varphi - \varphi_p)\ .$$

- It can be shown that $s_{00} = \sqrt{4\pi}$ and $s_{\ell m} = 0$ for $0 < \ell < 2L, \forall m$.
- Thus, the sampling distribution may be written

$$s(\theta, \varphi) = 1 + \sum_{\ell=2L}^{\infty} \sum_{m=-\ell}^{\ell} s_{\ell m}\, Y_{\ell m}(\theta, \varphi)\ .$$

- The harmonic coefficients of the product of the original band-limited function and the sampling distribution $f^s = f \cdot s$ are then given by

$$f^s_{\ell m} = \sum_{t=0}^{2L-1} \sum_{p=0}^{2L-1} q_{\rm DH}(\theta_t)\, f(\theta_t, \varphi_p)\, Y^*_{\ell m}(\theta_t, \varphi_p)\ ,$$

- Notice that these harmonic coefficients are given by the DH quadrature rule and it simply remains to prove that the harmonic coefficients of f^s agree with those of f for the harmonic range of interest (*i.e.* for $0 \leq \ell < L$).

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- We may write

$$f^s(\theta, \varphi) = f(\theta, \varphi) + \alpha(\theta, \varphi) \ ,$$

where

$$\alpha(\theta, \varphi) = \sum_{\ell=2L}^{\infty} \sum_{m=-\ell}^{\ell} s_{\ell m} \, Y_{\ell m}(\theta, \varphi) \sum_{\ell'=0}^{L-1} \sum_{m'=-\ell'}^{\ell'} f_{\ell' m'} \, Y_{\ell' m'}(\theta, \varphi) \ .$$

- Since the product of two spherical harmonic functions $Y_{\ell m}(\theta, \varphi) \, Y_{\ell' m'}(\theta, \varphi)$ can be written as a sum of spherical harmonics with minimum degree $|\ell - \ell'|$, the aliasing error $\alpha(\theta, \varphi)$ contains non-zero harmonic content for $\ell > L$ only.

- Aliasing is therefore outside of the harmonic range of interest and $f^s_{\ell m} = f_{\ell m}$ for $0 \leq \ell < L$, $|m| < \ell$, thus proving the exact quadrature rule.

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- Why $2L$ samples in θ? Recap...
- Stems from the implicit definition of the quadrature weights:

$$\sum_{t=0}^{N_\theta - 1} q_{\mathrm{DH}}(\theta_t)\, P_\ell(\cos\theta_t) = \frac{2\pi}{L}\,\delta_{\ell 0}\,, \quad \forall \ell < 2L\,.$$

- This is essentially an exact quadrature rule for the integration of Legendre polynomials, since

$$\int_0^\pi \mathrm{d}\theta \sin\theta\, P_\ell(\cos\theta_t)\, P_{\ell'}(\cos\theta_t) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} \quad \Rightarrow \quad \int_0^\pi \mathrm{d}\theta \sin\theta\, P_\ell(\cos\theta_t) = 2\,\delta_{\ell 0}\,.$$

- An exact quadrature rule is developed by appealing to the orthogonality of the complex exponentials on $[0, 2\pi)$:

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$$= \sum_{k=0}^{\lfloor \ell/2 \rfloor} \frac{2}{(2k+1)\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \underbrace{\sin\theta\, \sin((2k+1)\theta)\, P_\ell(\cos\theta_t)}_{\text{Trig. poly. of max degree } 2(\ell+1)}$$

- Require $4L$ samples in θ over $2\pi \Rightarrow$ $2L$ samples in θ on the sphere
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- Require $4L$ samples in θ over 2π $\Rightarrow$ $2L$ samples in θ on the sphere
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MW sampling theorem

- MW sampling theorem follows by a factoring of rotations and then by associating the sphere with the torus through a periodic extension.
- Similar (in flavour but not detail!) to making a periodic extension in θ of a function f on the sphere.
- Whereas DH perform an implicit extension of θ to $[0, 2\pi)$ in developing their exact quadrature rule, we perform an explicit extension of f to $[0, 2\pi)$ but restrict the continuous integral defining the quadrature weights to $[0, \pi]$.

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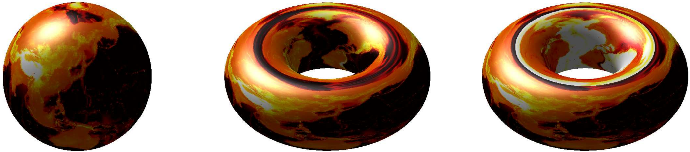

(a) Function on sphere (b) Even function on torus (c) Odd function on torus

Figure: Associating functions on the sphere and torus

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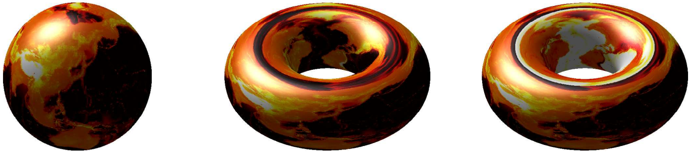

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MW sampling theorem

- By a factoring of rotations, a reordering of summations and a separation of variables, the forward transform of ${}_sf$ may be written:

$$ {}_sf_{\ell m} = (-1)^s \, \mathrm{i}^{m+s} \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'=-(L-1)}^{L-1} \Delta^{\ell}_{m' m} \, \Delta^{\ell}_{m', -s} \, {}_sG_{mm'} \ , $$

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- We develop an exact quadrature rule to evaluate the integral over θ by extending ${}_sG_m(\theta)$ to the domain $\theta \in [0, 2\pi)$ through the construction

$$ {}_s\tilde{G}_m(\theta_t) = \begin{cases} {}_sG_m(\theta_t)\ , & t \in \{0, 1, \dots, L-1\} \\ (-1)^{m+s}\ {}_sG_m(\theta_{2L-2-t})\ , & t \in \{L, \dots, 2L-2\} \end{cases} , $$

  so that ${}_s\tilde{G}_m(\theta_t)$ may be expressed by a Fourier series.

- Substituting into the integral over θ yields

$$ {}_sG_{mm'} = 2\pi \sum_{m''=-(L-1)}^{L-1} {}_sF_{mm''}\ w(m''-m')\ , $$

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$$ w(m') = \int_0^\pi \mathrm{d}\theta \sin\theta\ \mathrm{e}^{\mathrm{i}m'\theta} = \begin{cases} \pm \mathrm{i}\pi/2, & m' = \pm 1 \\ 0, & m' \text{ odd},\ m' \neq \pm 1 \\ 2/(1-m'^2), & m' \text{ even} \end{cases} , $$

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$$ {}_sF_{mm'} = \frac{1}{2\pi(2L-1)} \sum_{t=-(L-1)}^{L-1} {}_s\tilde{G}_m(\theta_t)\ \mathrm{e}^{-\mathrm{i}m'\theta_t} . $$

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Quadrature

- Sampling theorems effectively encode (often implicitly) an exact quadrature rule for evaluating the integral of a band-limited function on the sphere.

- The quadrature rule can be made explicit:

$$\int_{\mathrm{S}^2} \mathrm{d}\Omega(\theta, \varphi)\; {}_sf(\theta, \varphi) = \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} q(\theta_t)\; {}_sf(\theta_t, \varphi_p)\;,$$

  where N_θ, N_θ, $q \in \{q_{\mathrm{DH}},\ q_{\mathrm{MW}}\}$ and the sample positions $\{\theta_t, \varphi_p\}$ depend on the chosen sampling theorem.

Comparison

	DH Divide-and-conquer	DH Semi-naive	MW
Pixelisation scheme	equiangular	equiangular	equiangular
Asymptotic complexity	$\mathcal{O}(L^{5/2} \log_2^{1/2} L)$	$\mathcal{O}(L^3)$	$\mathcal{O}(L^3)$
Precomputation	Y	N	N
Stability	N	Y	Y
Flexibility of Wigner recursion	N	N	Y
Number of samples	$\mathbf{4L^2}$	$\mathbf{4L^2}$	$\mathbf{2L^2}$

Comparison

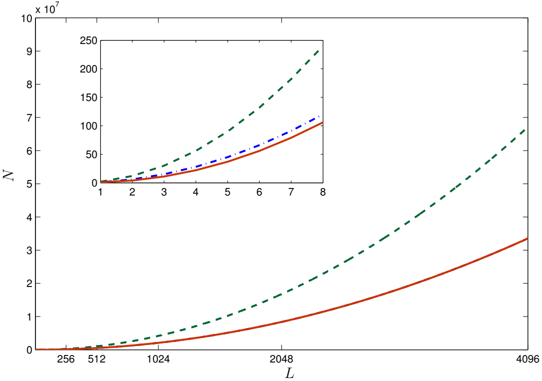

Figure: Number of samples (MW=red; DH=green; GL=blue)

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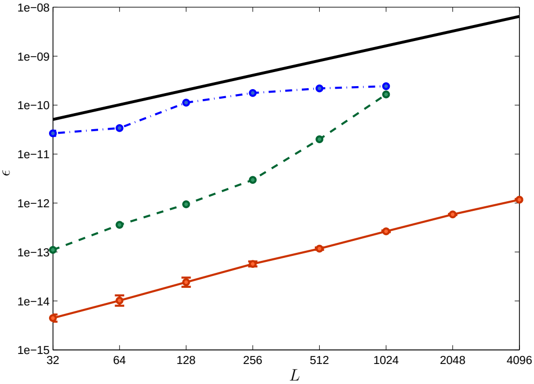

Figure: Numerical accuracy (MW=red; DH=green; GL=blue)

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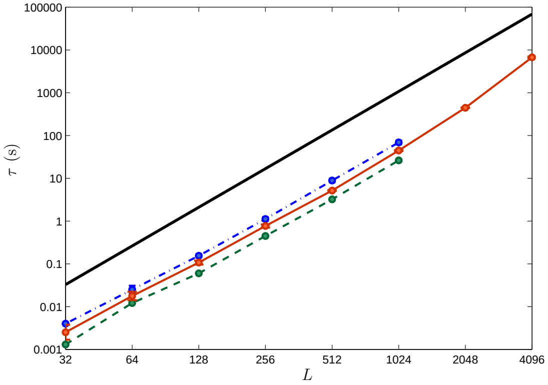

Figure: Computation time(MW=red; DH=green; GL=blue)

Compressive sensing on the sphere

- A reduction in the number of samples required to represent a band-limited signal on the sphere has important implications for compressive sensing.
- Many natural signals are sparse in measures defined in the spatial domain, such as in the magnitude of their gradient.
- A more efficient sampling of a band-limited signal on the sphere improves both the dimensionality and sparsity of the signal in the spatial domain.
- For a given number of measurements, a more efficient sampling theorem improves the quality of compressive sampling reconstruction.
- Illustrate with a total variation (TV) inpainting problem on the sphere.

TV inpainting

- Consider inpainting problem $y = \Phi x + n$ in the context of different sampling theorems, where:
  - the samples of f are denoted by the concatenated vector $x \in \mathbb{R}^N$;
  - N is the number of samples on the sphere of the chosen sampling theorem;
  - M noisy measurements $y \in \mathbb{R}^M$ are acquired;
  - the measurement operator $\Phi \in \mathbb{R}^{M \times N}$ represents a random masking of the signal;
  - the noise $n \in \mathbb{R}^M$ is assumed to be iid Gaussian with zero mean.
- Define TV norm on the sphere:

$$\int_{\mathbb{S}^2} \mathrm{d}\Omega\, |\nabla f| \simeq \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} |\nabla f|\, q(\theta_t) \simeq \sum_{t=0}^{N_\theta - 1} \sum_{p=0}^{N_\varphi - 1} \sqrt{q^2(\theta_t)(\delta_\theta x)^2 + \frac{q^2(\theta_t)}{\sin^2\theta_t}(\delta_\varphi x)^2} \equiv \|x\|_{\mathrm{TV}} .$$

- TV inpainting problem solved directly on the sphere:

$$x^\star = \arg\min_{x} \|x\|_{\mathrm{TV}} \ \text{such that} \ \|y - \Phi x\|_2 \leq \epsilon .$$

- TV inpainting problem solved in harmonic space:

$$\hat{x}^\star = \arg\min_{\hat{x}} \|\Lambda \hat{x}\|_{\mathrm{TV}} \ \text{such that} \ \|y - \Phi \Lambda \hat{x}\|_2 \leq \epsilon ,$$

where Λ represents the inverse spherical harmonic transform and harmonic coefficients are represented by the concatenated vector $\hat{x} \in \mathbb{C}^{L^2}$.

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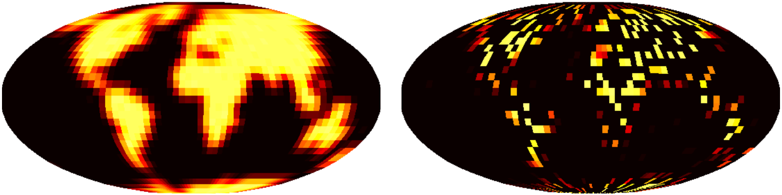
TV inpainting: low-resolution simulations

- Solve TV inpainting problem on the sphere in the context of the Driscoll & Healy sampling theorem and our new sampling theorem.



(a) Ground truth (b) Measurements

Figure: Earth topographic data reconstructed in the harmonic domain for $M/L^2 = 1/2$

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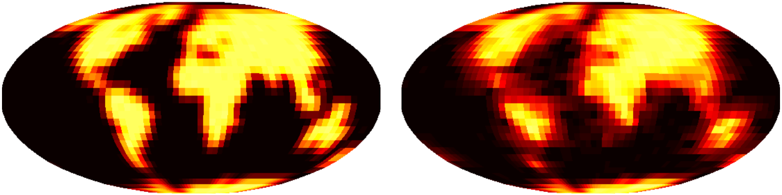

(a) Ground truth (b) DH reconstruction

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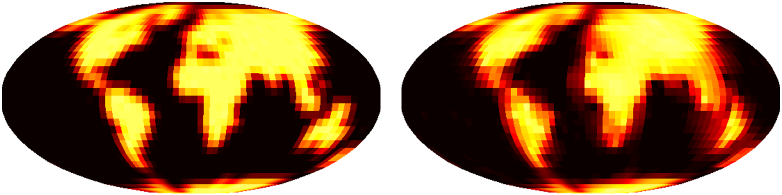

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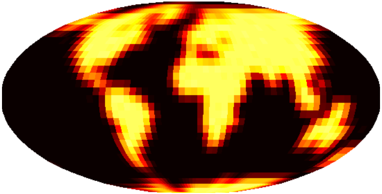

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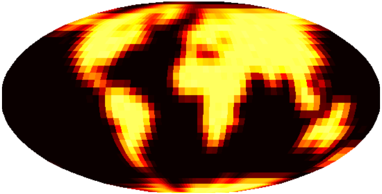

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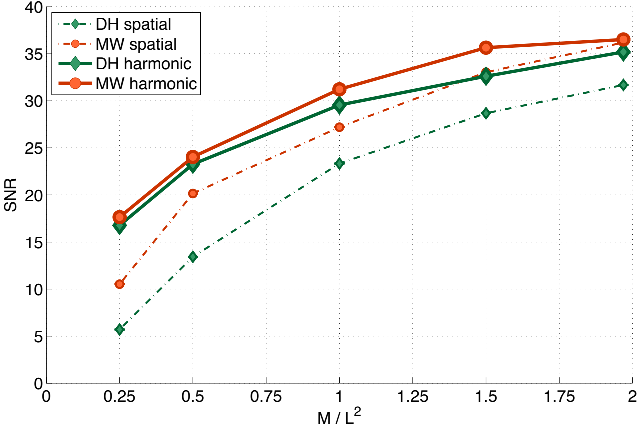

Figure: Reconstruction performance for the DH and MW sampling theorems

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- Require fast adjoint operators as well as fast spherical harmonic transforms to solve the optimisation problems.
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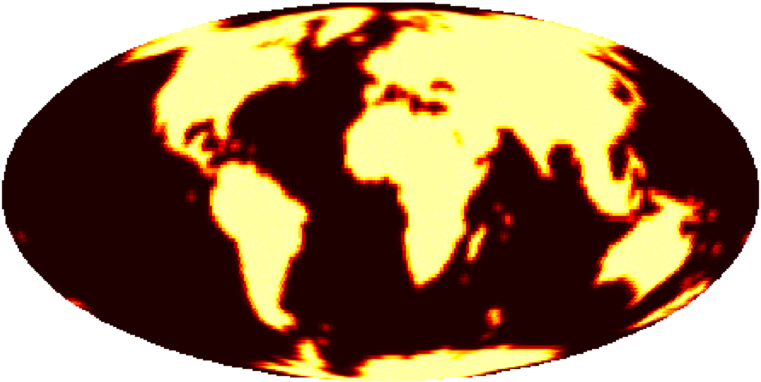

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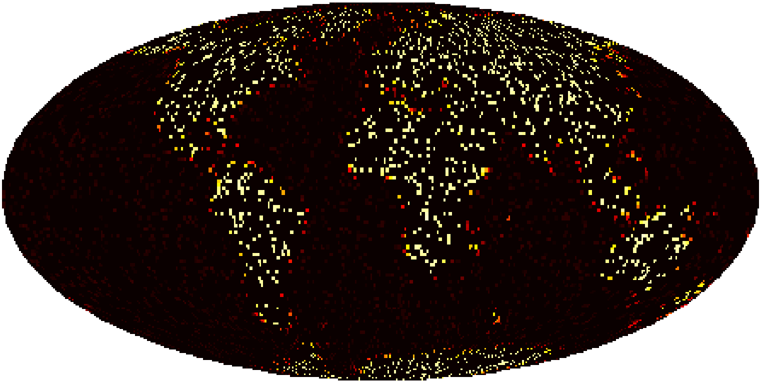

Figure: Measurements ($M/L^2 = 1/4$)

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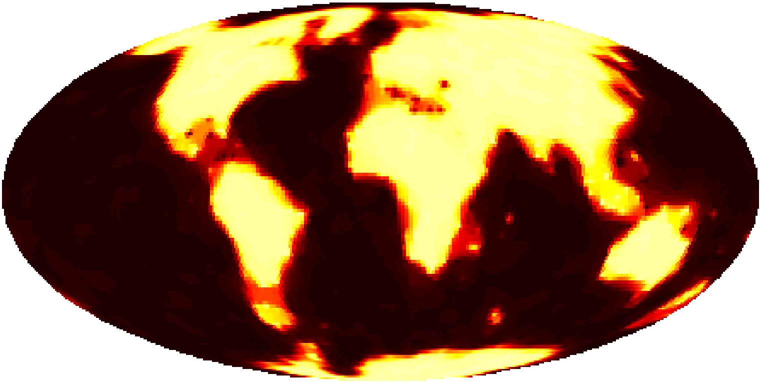

Figure: MW reconstruction ($M/L^2 = 1/4$)

Summary

- We have developed a new sampling theorem on the sphere requiring fewer than half the number of samples of the canonical Driscoll & Healy sampling theorem.
- A reduction in the number of samples required to represent a band-limited signal on the sphere has important implications for compressive sensing, both in terms of the dimensionality and sparsity of signals.
- We have demonstrated improved reconstruction quality when solving an inpainting problem in the context of different sampling theorems.
- We have developed fast adjoint spherical harmonic transform operators to tackle problems with high band-limits.

Related publications

- McEwen, J. D. and Wiaux, Y., *A novel sampling theorem on the sphere*, IEEE Trans. Sig. Proc., 59(12): 5876-5887, 2011.
- McEwen, J. D., Puy, G., Thiran, J.-P., Vandergheynst, P., Ville, D. V. D., and Wiaux, Y., *Efficient and compressive sampling on the sphere*, IEEE Trans. Sig. Proc., submitted, 2011.

SSHT code

- Code available to compute exact spin spherical harmonic transforms (SSHT) in the context of our new sampling theorem: `http://www.jasonmcewen.org/`