Revisiting the Spread Spectrum Effect

via a sparse variant of the w-projection algorithm

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Three pillars of compressive sensing

1. High-fidelity imaging
2. Advanced algorithms
3. Theory

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Outline

1. Preliminaries
2. Spread spectrum effect
3. Sparse w-projection
4. Results
5. Conclusions & outlook

Preliminaries

Radio interferometric measurement equation

- The complex visibility measured by an interferometer is given by

$$y(\boldsymbol{u}, w) = \int_{D^2} A(\boldsymbol{l})\, x(\boldsymbol{l})\, C(\|\boldsymbol{l}\|_2)\, \mathrm{e}^{-\mathrm{i}2\pi \boldsymbol{u}\cdot\boldsymbol{l}}\, \frac{\mathrm{d}^2\boldsymbol{l}}{n(\boldsymbol{l})},$$

visibilities

where the w-modulation $C(\|\boldsymbol{l}\|_2)$ is given by

$$C(\|\boldsymbol{l}\|_2) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}.$$

w-modulation

- Various assumptions are often made regarding the size of the field-of-view (FoV):
  - Small-field with $\|\boldsymbol{l}\|^2 w \ll 1$ $\Rightarrow$ $C(\|\boldsymbol{l}\|_2) \simeq 1$
  - Small-field with $\|\boldsymbol{l}\|^4 w \ll 1$ $\Rightarrow$ $C(\|\boldsymbol{l}\|_2) \simeq \mathrm{e}^{\mathrm{i}\pi w \|\boldsymbol{l}\|^2}$
  - Wide-field $\Rightarrow$ $C(\|\boldsymbol{l}\|_2) = \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-\|\boldsymbol{l}\|^2}\right)}$

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Radio interferometric inverse problem

- Consider the ill-posed inverse problem of radio interferometric imaging:

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n},$$

  where $\boldsymbol{y}$ are the measured visibilities, Φ is the linear measurement operator, $\boldsymbol{x}$ is the underlying image and $\boldsymbol{n}$ is instrumental noise.

- Measurement operator $\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A}$ may incorporate:
  - primary beam $\mathbf{A}$ of the telescope;
  - w-modulation modulation $\mathbf{C}$;
  - Fourier transform $\mathbf{F}$;
  - masking $\mathbf{M}$ which encodes the incomplete measurements taken by the interferometer.

- Compressive sensing imaging solves sparse optimisations problems:

$$\alpha^\star = \arg\min_{\alpha} \|\alpha\|_1 \text{ such that } \|y - \Phi\Psi\alpha\|_2 \leq \epsilon \qquad \text{BPDN}$$

$$\min_{\bar{x}\in\mathbb{R}^N} \|W\Psi^T\bar{x}\|_1 \quad \text{subject to} \quad \|y - \Phi\bar{x}\|_2 \leq \epsilon \quad \text{and} \quad \bar{x} \geq 0 \qquad \text{SARA}$$

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Sparsity and coherence

- What drives the quality of compressive sensing reconstruction?
- Number of measurements M required to achieve exact reconstruction given by

$$M \geq c\mu^2 K \log N ,$$

where K is the sparsity and N the dimensionality.

- Coherence between the measurement vectors and atoms of sparsity dictionary given by

$$\mu = \sqrt{N} \max_{i,j} |\langle \Psi_i, \Phi_j \rangle| .$$

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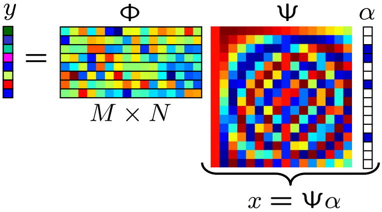

Spread spectrum effect
Review

- Non-coplanar baselines and wide fields $\rightarrow$ w-modulation $\rightarrow$ spread spectrum effect (first considered by Wiaux *et al.* 2009b).
- Recall, w-modulation operator $\mathbf{C}$ has elements defined by

$$C(l,m) \equiv \mathrm{e}^{\mathrm{i}2\pi w\left(1-\sqrt{1-l^2-m^2}\right)} \simeq \mathrm{e}^{\mathrm{i}\pi w\|\boldsymbol{l}\|^2} \quad \text{for} \quad \|\boldsymbol{l}\|^4\, w \ll 1 ,$$

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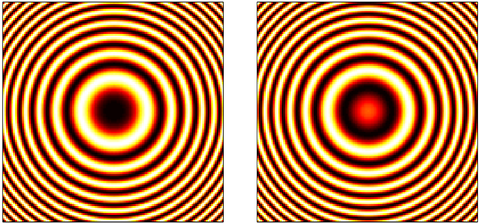

(a) Real part (b) Imaginary part

Figure: Chirp modulation.

Spread spectrum effect
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Spread spectrum effect in a nutshell

1. Radio interferometers take (essentially) Fourier measurements.
2. Recall, the coherence is the maximum inner product between measurement vectors and sparsifying atoms.
3. Thus, coherence is (essentially) the maximum of the Fourier coefficients of the atoms of the sparsifying dictionary.
4. w-modulation spreads the spectrum of the atoms of the sparsifying dictionary, reducing the maximum Fourier coefficient.
5. Spreading the spectrum reduces coherence, thus improving reconstruction fidelity.

- Improved reconstruction fidelity of the spread spectrum effect demonstrated with simulations by Wiaux *et al.* (2009b) with constant w (for simplicity).
- Here we study the spread spectrum effect for varying w, realistic images and various sparse representations.

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Sparse w-projection

- Apply the w-projection algorithm (Cornwell *et al.* 2008) to shift the w-modulation through the Fourier transform:

$$\boxed{\Phi = \mathbf{M}\,\mathbf{F}\,\mathbf{C}\,\mathbf{A} \quad \Rightarrow \quad \Phi = \hat{\mathbf{C}}\,\mathbf{F}\,\mathbf{A}} \; .$$

- Naively, expressing the application of the w-modulation in this manner is computationally less efficient that the original formulation but it has two important advantages.
- Different w for each (u, v), while still exploiting FFT.
- Many of the elements of $\hat{\mathbf{C}}$ will be close to zero.

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Sparse w-projection

Sparsity of w-modulation kernel in Fourier space

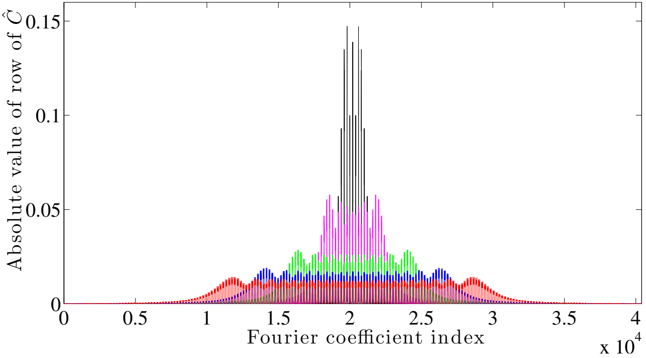

Figure: Rows of Fourier representation of w-modulation operator $\hat{\mathbf{C}}$.

Sparse w-projection

Dynamic sparsification

- We make a sparse matrix approximation of $\hat{\mathrm{C}}$ to speed up its computational application and reduce memory requirements.
- Sparsify $\hat{\mathrm{C}}$ by dynamic thresholding.
- Retain E% of the energy content for each visibility measurement.
- Support of w-modulation kernel in Fourier space determined dynamically, so don't require any prior information about structure.
- Generic procedure applicable for any direction-dependent effect (DDE).

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Sparsified w-modulation kernels

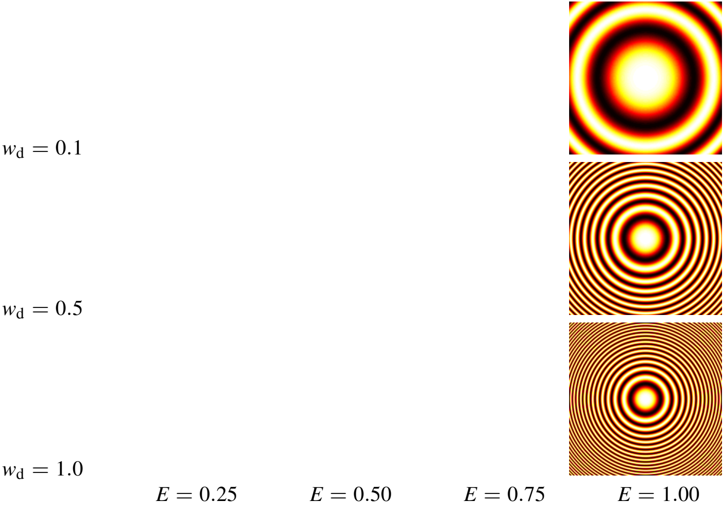

Figure: w-modulation kernel.

Sparse w-projection

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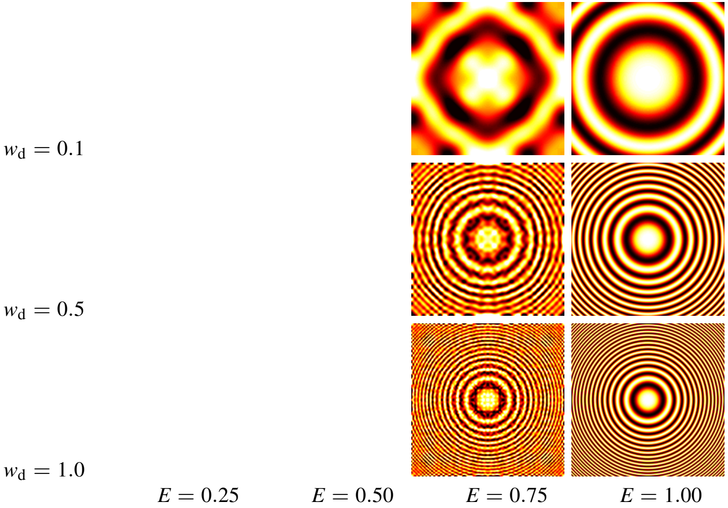

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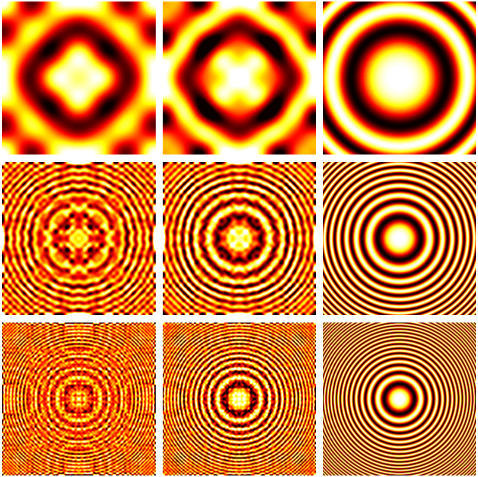

$w_d = 0.1$

$w_d = 0.5$

$w_d = 1.0$

$E = 0.25$ $E = 0.50$ $E = 0.75$ $E = 1.00$

Figure: w-modulation kernel.

Sparse w-projection

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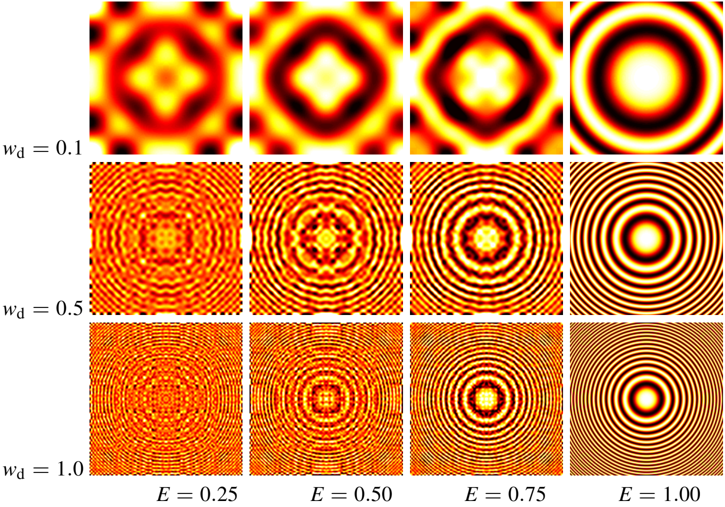

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Sparse w-projection
Proportion of non-zero entries

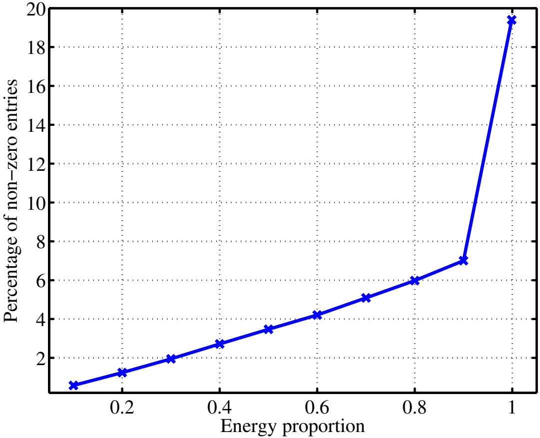


Figure: Percentage of non-zero entries as a function of preserved energy proportion.

Sparse w-projection

Runtime improvements

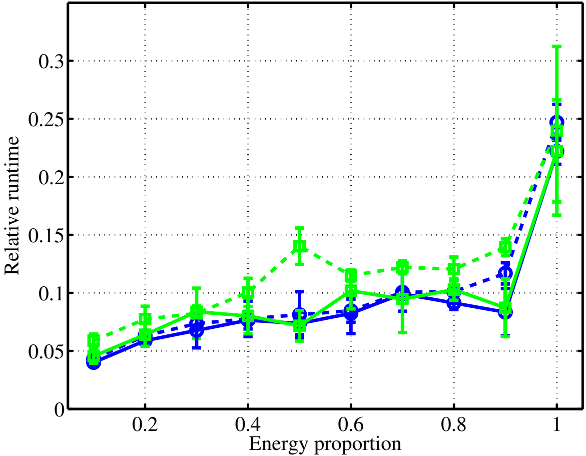


Figure: Relative runtime as a function of preserved energy proportion for 10% (dashed) and 50% (solid) visibility coverages.

Sparse w-projection

Impact on reconstruction quality

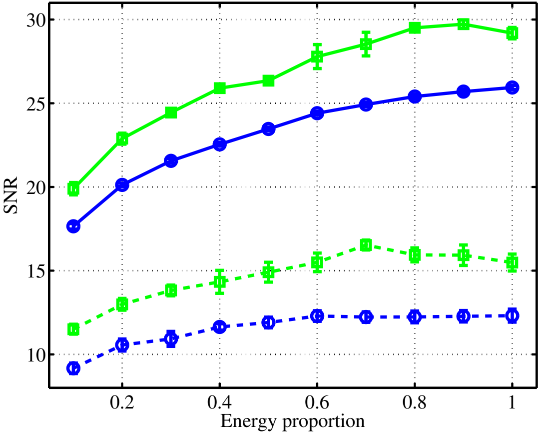
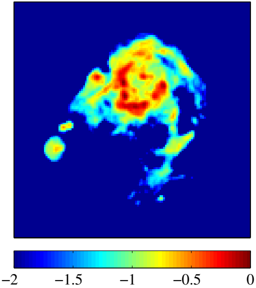


Figure: Reconstruction quality as a function of preserved energy proportion for 10% (dashed) and 50% (solid) visibility coverages.

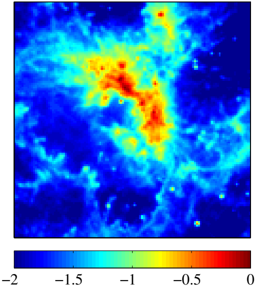
Results

Ground truth for simulations

- Perform simulations to assess the effectiveness of the spread spectrum effect in the presence of varying w.
- Consider idealised simulations.



(a) HII region in M31

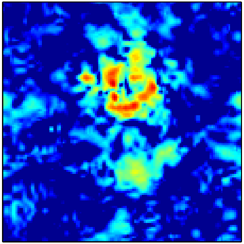


(b) 30 Doradus (30Dor)

Figure: Ground truth images in logarithmic scale.

Results

Reconstructed images



(a) $w_{\rm d} = 0 \rightarrow$ SNR$= 5$dB

Figure: Reconstructed images of M31 for 10% coverage.

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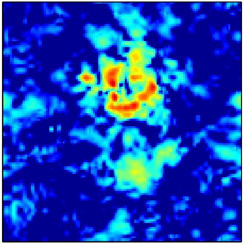

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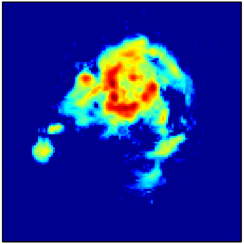

(c) $w_{\rm d} = 1 \rightarrow$ SNR$= 19$dB

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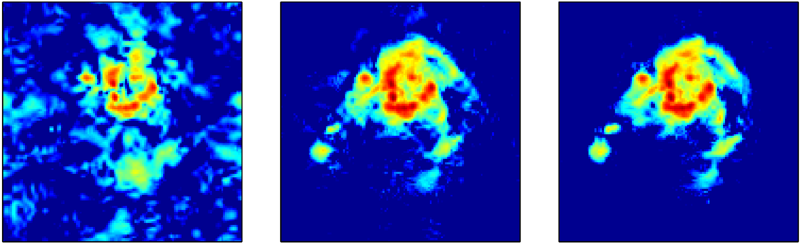
Results

Reconstructed images



(a) $w_{\mathrm{d}} = 0 \rightarrow$ SNR= 5dB (b) $w_{\mathrm{d}} \sim \mathcal{U}(0, 1) \rightarrow$ SNR= 16dB (c) $w_{\mathrm{d}} = 1 \rightarrow$ SNR= 19dB

Figure: Reconstructed images of M31 for 10% coverage.

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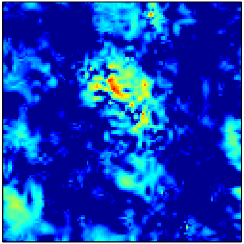
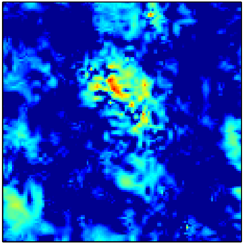


(a) $w_{\rm d} = 0 \rightarrow$ SNR$=$ 2dB

Figure: Reconstructed images of 30Dor for 10% coverage.

Results

Reconstructed images



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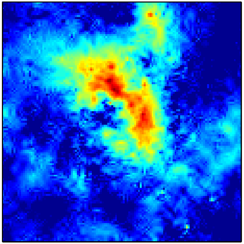

(c) $w_{\rm d} = 1 \rightarrow$ SNR$= 15$dB

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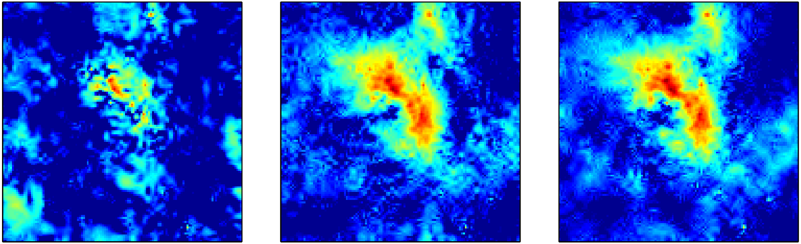

(a) $w_{\rm d} = 0 \rightarrow$ SNR$=$ 2dB (b) $w_{\rm d} \sim \mathcal{U}(0, 1) \rightarrow$ SNR$=$ 12dB (c) $w_{\rm d} = 1 \rightarrow$ SNR$=$ 15dB

Figure: Reconstructed images of 30Dor for 10% coverage.

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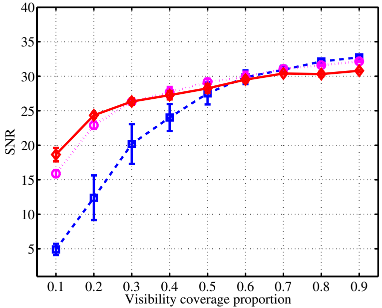
Reconstruction performance



(a) Daubechies 8 (Db8) wavelets

Figure: Reconstruction fidelity for M31.

Improvement in reconstruction fidelity due to the spread spectrum effect for varying w is almost as large as the case of constant maximum w!

- As expected, for the case where coherence is already optimal, there is little improvement.

Results

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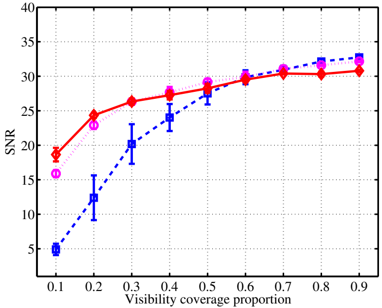

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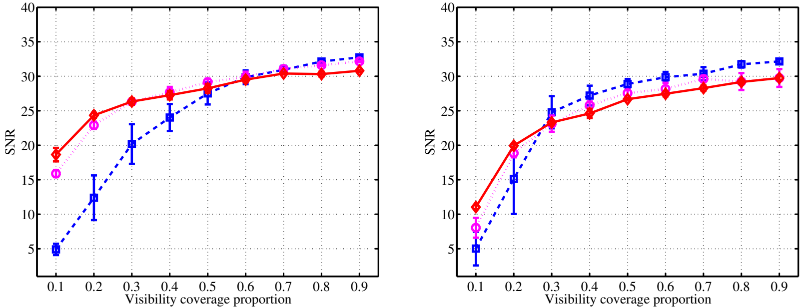

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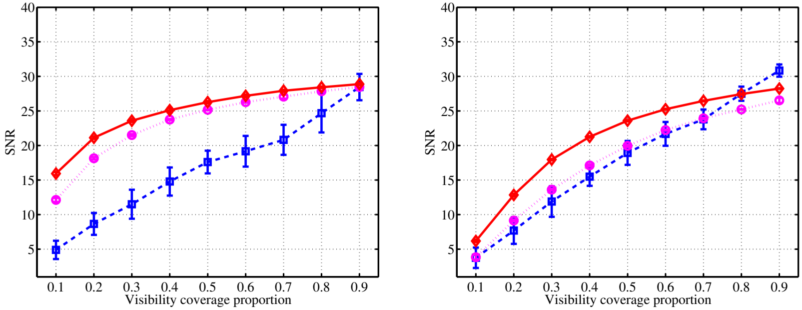

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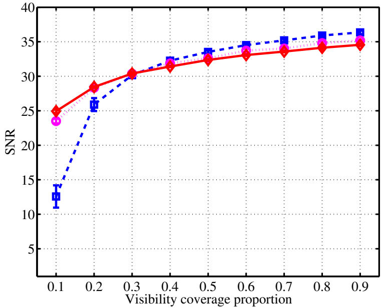

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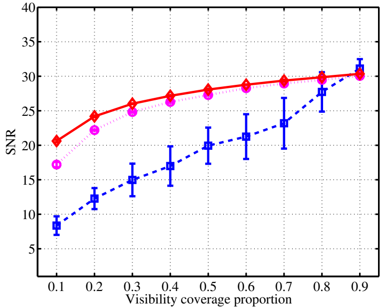

(b) 30 Dor

Figure: Reconstruction fidelity using SARA.

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Conclusions & outlook

Conclusions

- If the non-coplanar baseline and wide FoV setting is modeled accurately, then due to the spread spectrum effect...
- ...the same image reconstruction quality can be achieved with considerably fewer baselines
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> Optimise future telescope configurations to promote large w-components
> $\rightarrow$ enhance the spread spectrum effect
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- We have just released the PURIFY code to scale to the realistic setting.
- Includes state-of-the-art convex optimisation algorithms implemented in C.
- Integration with CASA is in progress and should be complete soon.
- Plan to perform more extensive comparisons with traditional techniques, such as CLEAN, MS-CLEAN and MEM.

Encourage you to apply PURIFY to your real observational data.

PURIFY code http://basp-group.github.io/purify/

Next-generation radio interferometric imaging
Carrillo, McEwen, Wiaux

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Extra Slides

Compressive sensing

> “Nothing short of revolutionary.”

– National Science Foundation

- Developed by Emmanuel Candes and David Donoho (and others).

(a) Emmanuel Candes (b) David Donoho

Compressive sensing

- Next evolution of wavelet analysis $\rightarrow$ wavelets are a key ingredient.
- The mystery of JPEG compression (discrete cosine transform; wavelet transform).
- Move compression to the acquisition stage $\rightarrow$ compressive sensing.
- Acquisition versus imaging.

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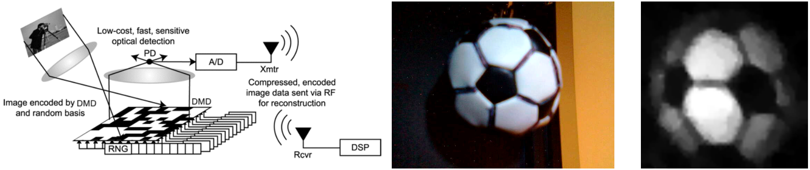

(a) Architecture (b) Scene (c) Recov. (20% meas.)

Figure: Single pixel camera

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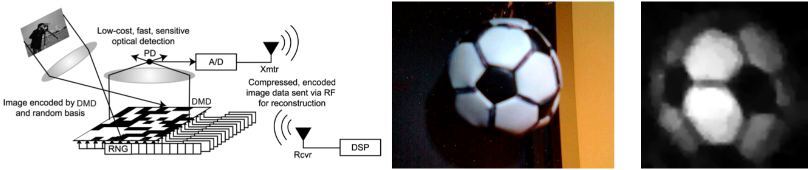

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An introduction to compressive sensing

Operator description

- Linear operator (linear algebra) representation of signal decomposition:

$$x(t) = \sum_i \alpha_i \Psi_i(t) \quad \rightarrow \quad \boldsymbol{x} = \sum_i \Psi_i \alpha_i = \begin{pmatrix} | \\ \Psi_0 \\ | \end{pmatrix} \alpha_0 + \begin{pmatrix} | \\ \Psi_1 \\ | \end{pmatrix} \alpha_1 + \cdots \quad \rightarrow \quad \boxed{\boldsymbol{x} = \Psi \boldsymbol{\alpha}}$$

- Linear operator (linear algebra) representation of measurement:

$$y_i = \langle x, \Phi_j \rangle \quad \rightarrow \quad \boldsymbol{y} = \begin{pmatrix} - \; \Phi_0 \; - \\ - \; \Phi_1 \; - \\ \vdots \end{pmatrix} \boldsymbol{x} \quad \rightarrow \quad \boxed{\boldsymbol{y} = \Phi \boldsymbol{x}}$$

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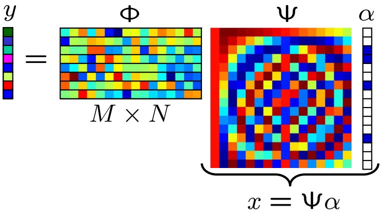

An introduction to compressive sensing

Promoting sparsity via ℓ_1 minimisation

- Ill-posed inverse problem:

$$\boldsymbol{y} = \Phi \boldsymbol{x} + \boldsymbol{n} = \Phi \Psi \boldsymbol{\alpha} + \boldsymbol{n} .$$

- Recall norms given by:

$$\|\alpha\|_0 = \text{no. non-zero elements} \qquad \|\alpha\|_1 = \sum_i |\alpha_i| \qquad \|\alpha\|_2 = \Big(\sum_i |\alpha_i|^2\Big)^{1/2}$$

- Solve by imposing a regularising prior that the signal to be recovered is sparse in Ψ, *i.e.* solve the following ℓ_0 optimisation problem:

$$\boldsymbol{\alpha}^\star = \arg\min_{\boldsymbol{\alpha}} \|\alpha\|_0 \text{ such that } \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon ,$$

where the signal is synthesising by $\boldsymbol{x}^\star = \Psi \boldsymbol{\alpha}^\star$.

- Solving this problem is difficult (combinatorial).

- Instead, solve the ℓ_1 optimisation problem (convex):

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- Solutions of the ℓ_0 and ℓ_1 problems are often the same.
- Restricted isometry property (RIP):

$$(1-\delta_K)\|\boldsymbol{\alpha}\|_2^2 \leq \|\Theta\boldsymbol{\alpha}\|_2^2 \leq (1+\delta_K)\|\boldsymbol{\alpha}\|_2^2 ,$$

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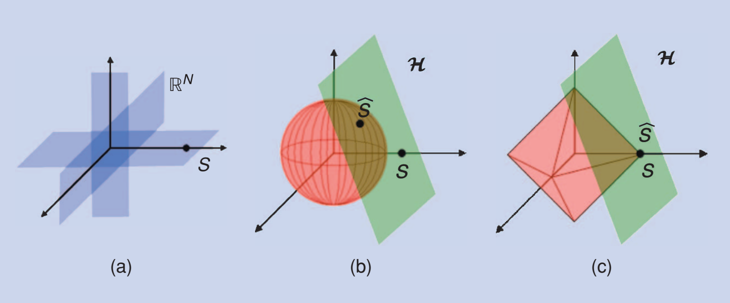

Figure: Geometry of (a) ℓ_0 (b) ℓ_2 and (c) ℓ_1 problems. [Credit: Baraniuk (2007)]

An introduction to compressive sensing
Coherence

- In the absence of noise, compressed sensing is exact!
- Number of measurements required to achieve exact reconstruction is given by

$$M \geq c\mu^2 K \log N ,$$

where K is the sparsity and N the dimensionality.

- The coherence between the measurement and sparsity basis is given by

$$\mu = \sqrt{N} \max_{i,j} |\langle \Psi_i, \Phi_j \rangle| .$$

- Robust to noise.

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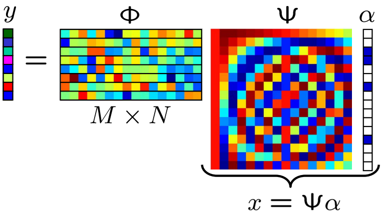

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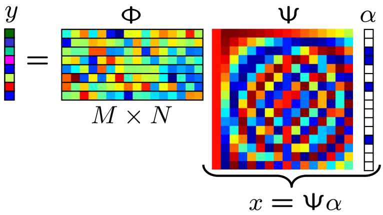

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Analysis vs synthesis

- Many new developments (*e.g.* analysis vs synthesis, cosparsity, structured sparsity).
- Synthesis-based framework:

$$\boldsymbol{\alpha}^\star = \underset{\boldsymbol{\alpha}}{\arg\min} \|\boldsymbol{\alpha}\|_1 \text{ such that } \|\boldsymbol{y} - \Phi\Psi\boldsymbol{\alpha}\|_2 \leq \epsilon \,.$$

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  where the signal $\boldsymbol{x}^\star$ is recovered directly.
- Concatenating dictionaries (Rauhut *et al.* 2008) and sparsity averaging (Carrillo, McEwen & Wiaux 2013)

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