Fourier-Laguerre transform, convolution and wavelets on the ball

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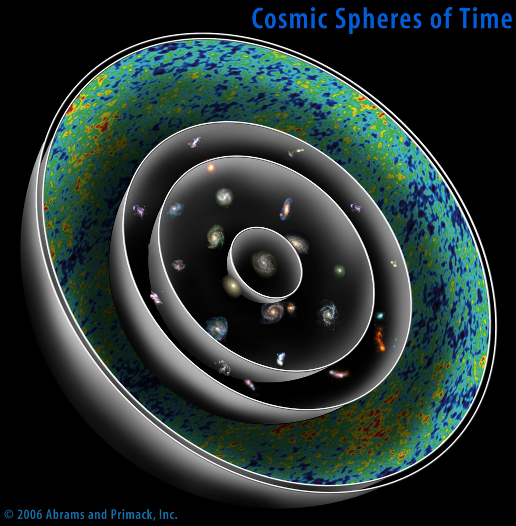
Cosmic history



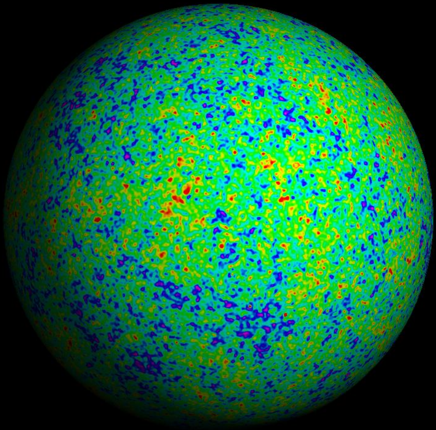
From the CMB to today

Credit: WMAP

Large-scale structure (LSS) of the Universe

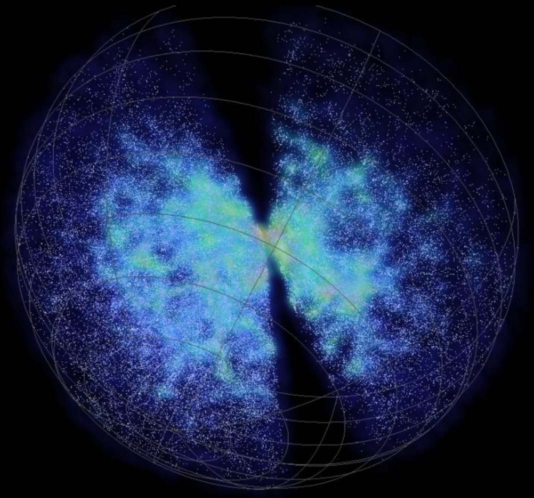
Credit: Teyssier

Cosmic data-sets: Cosmic microwave background (CMB)



Credit: WMAP

Cosmic data-sets: Galaxy surveys tracing the LSS



Credit: SDSS

Outline

1. Fourier-Laguerre transform
2. Fourier-Laguerre convolution
3. Fourier-Laguerre wavelets
4. Illustration

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Fourier-Bessel transform on the ball

- Consider functions on the ball $\mathbb{B}^3 = \mathbb{R}^+ \times \mathbb{S}^2$, *i.e.* $f \in \mathrm{L}^2(\mathbb{B}^3)$.
- Fourier-Bessel functions are the canonical orthogonal basis on the ball since they are the eigenfunctions of the Laplacian:

$$X_{\ell m}(k, \boldsymbol{r}) = j_\ell(kr) Y_{\ell m}(\theta, \varphi).$$

with spherical coordinates $\boldsymbol{r} = (r, \theta, \varphi) \in \mathbb{B}^3$, where $r \in \mathbb{R}^+$ denotes radius, $\theta \in [0, \pi]$ colatitude and $\varphi \in [0, 2\pi)$ longitude, and where $k \in \mathbb{R}^+$, $\ell \in \mathbb{N}$, $m \in \mathbb{Z}$, $|m| \le \ell$.

- Fourier-Bessel transform of $f \in \mathrm{L}^2(\mathbb{B}^3)$ reads

$$\tilde{f}_{\ell m}(k) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{B}^3} \mathrm{d}^3\boldsymbol{r}\, f(\boldsymbol{r})\, j_\ell^*(kr)\, Y_{\ell m}^*(\theta, \varphi),$$

where $\mathrm{d}^3\boldsymbol{r} = r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi$ is the usual measure in spherical coordinates.

- Inverse transform given by

$$f(\boldsymbol{r}) = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{\mathbb{R}^+} \mathrm{d}k k^2\, \tilde{f}_{\ell m}(k)\, j_\ell(kr)\, Y_{\ell m}(\theta, \varphi).$$

- But… does not admit an applicable sampling theorem.

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Fourier-Laguerre transform on the ball

- Define the Fourier-Laguerre basis functions by $Z_{\ell mp}(\boldsymbol{r}) = K_p(r) Y_{\ell m}(\theta, \varphi)$.
- Radial basis functions defined by the spherical Laguerre functions

$$K_p(r) \equiv \sqrt{\frac{p!}{(p+2)!}} \frac{e^{-r/2\tau}}{\sqrt{\tau^3}} L_p^{(2)}\left(\frac{r}{\tau}\right) ,$$

where $L_p^{(2)}$ is the p-th generalised Laguerre polynomial of order two.

- A signal $f \in \mathrm{L}^2(\mathbb{B}^3)$ can then be decomposed as

$$f(\boldsymbol{r}) = \sum_{p=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell mp} \, Z_{\ell mp}(\boldsymbol{r}),$$

where the harmonic coefficients are given by the usual projection

$$f_{\ell mp} = \langle f | Z_{\ell mp} \rangle_{\mathbb{B}^3} = \int_{\mathbb{B}^3} \mathrm{d}^3\boldsymbol{r} \, f(\boldsymbol{r}) \, Z^*_{\ell mp}(\boldsymbol{r}).$$

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Exact and fast Fourier-Laguerre transform

- Appeal to Gaussian quadrature to perform radial integral exactly.
- Appeal to new sampling theorem on the sphere (McEwen & Wiaux 2011) for angular part, which reduces the Nyquist rate on the sphere by a factor of two for equiangular sampling compared to the canonical approach (Driscoll & Healy 1994).
- Fast algorithms via separation of variables and factoring of rotations.
- $\Rightarrow$ Exact and fast Fourier-Laguerre transform.

- Separate the radial and angular components.
- Retain contact with the Fourier-Bessel coefficients.

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Accuracy of the Fourier-Laguerre transform

- For a band-limited signal, we can compute the Fourier-Laguerre transform exactly.

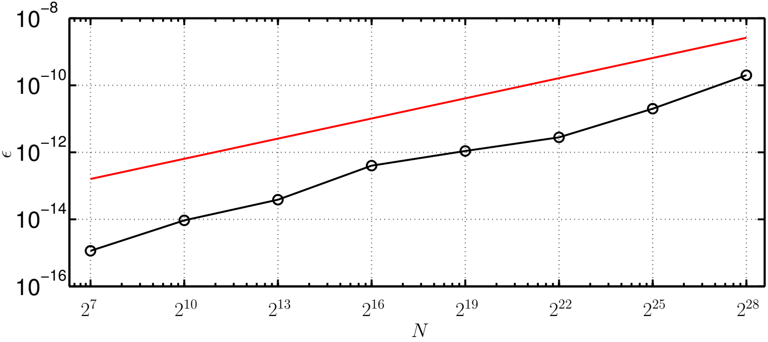


Figure: Numerical accuracy of Fourier-Laguerre transform

Computation time of the Fourier-Laguerre transform

- Fast algorithms to compute the Fourier-Laguerre transform.

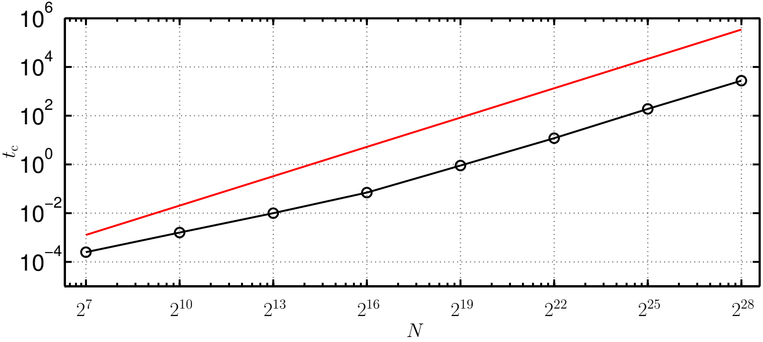
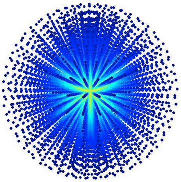


Figure: Computation time of Fourier-Laguerre transform

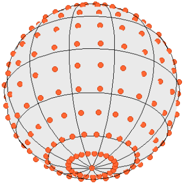
Codes to compute harmonic transforms



FLAG code: Fourier-Laguerre transforms

`http://www.flaglets.org`

Exact wavelets on the ball
Leistedt & McEwen (2012)



SSHT code: Spin spherical harmonic transforms

`http://www.ssht.org.uk`

A novel sampling theorem on the sphere
McEwen & Wiaux (2011)

Outline

1 Fourier-Laguerre transform

2 Fourier-Laguerre convolution

3 Fourier-Laguerre wavelets

4 Illustration

Translation and convolution on the radial line

- We construct translation and convolution operators on the radial line by analogy with the infinite line.
- For the standard orthogonal basis $\phi_\omega(x) = \mathrm{e}^{\mathrm{i}\omega x}$ translation of the basis functions defined by the shift of coordinates:

$$(\mathcal{T}_u^{\mathbb{R}} \phi_\omega)(x) \equiv \phi_\omega(x-u) = \phi_\omega^*(u)\phi_\omega(x) \ .$$

- Define translation of the spherical Laguerre basis functions on the radial line by analogy:

$$(\mathcal{T}_s K_p)(r) \equiv K_p(s) K_p(r) \ .$$

- Convolution on the radial line defined by

$$(f \star h)(r) \equiv \langle f | \mathcal{T}_r h \rangle_{\mathbb{R}^+} = \int_{\mathbb{R}^+} \mathrm{d}s s^2 \, f(s) \, (\mathcal{T}_r h)(s),$$

- In harmonic space, radial convolution is given by the product

$$(f \star h)_p = \langle f \star h | K_p \rangle_{\mathbb{R}^+} = f_p \, h_p \ .$$

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Translation and convolution on the radial line

- Translation on the radial line corresponds to convolution with the Dirac delta:

$$(f \star \delta_s)(r) = \sum_{p=0}^{\infty} f_p K_p(s) K_p(r) = (\mathcal{T}_s f)(r) \ .$$

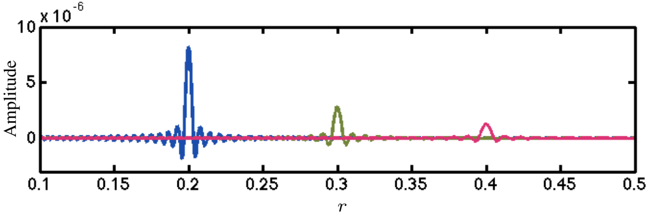

Figure: Band limited translated Dirac delta functions

Translation and convolution on the sphere

- Translation operator on the sphere is given by the standard three-dimensional rotation:

$$(\mathcal{R}_{(\alpha,\beta,\gamma)} h)(\theta,\varphi) = h(\mathcal{R}^{-1}_{(\alpha,\beta,\gamma)}(\theta,\varphi)),$$

  with $(\alpha,\beta,\gamma) \in \mathrm{SO}(3)$, where $\alpha \in [0, 2\pi)$, $\beta \in [0,\pi]$ and $\gamma \in [0, 2\pi)$.

- We make the association $\theta = \beta$ and $\varphi = \alpha$, *i.e.* $\mathcal{R}_{(\theta,\varphi)} \equiv \mathcal{R}_{(\alpha,\beta,0)}$, and restrict our attention to convolution with axisymmetric kernels that are invariant under azimuthal rotation, *i.e.* $\mathcal{R}_{(0,0,\gamma)} h = h$.

- Convolution on the sphere of $f \in \mathrm{L}^2(\mathbb{S}^2)$ with an axisymmetric kernel $h \in \mathrm{L}^2(\mathbb{S}^2)$ is given by

$$(f \star h)(\theta,\varphi) \equiv \langle f | \mathcal{R}_{(\theta,\varphi)} h \rangle_{\mathbb{S}^2} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\theta',\varphi')\, f(\theta',\varphi')\, \left(\mathcal{R}_{(\theta,\varphi)} h\right)^* (\theta',\varphi').$$

- In harmonic space, axisymmetric convolution may be written

$$(f \star h)_{\ell m} = \langle f \star h | Y_{\ell m} \rangle_{\mathbb{S}^2} = \sqrt{\frac{4\pi}{2\ell+1}}\, f_{\ell m}\, h^*_{\ell 0},$$

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Fourier-Laguerre translation and convolution

- Translation operator on the ball defined by combining the angular and radial translation operators, giving

$$\mathcal{T}_{\boldsymbol{r}} \equiv \mathcal{T}_r \mathcal{R}_{(\theta,\varphi)} .$$

- Convolution on the ball of $f \in \mathrm{L}^2(\mathbb{B}^3)$ with an axisymmetric kernel $h \in \mathrm{L}^2(\mathbb{B}^3)$ is defined by

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$$(f \star h)_{\ell m p} = \langle f \star h | Z_{\ell m p} \rangle_{\mathbb{B}^3} = \sqrt{\frac{4\pi}{2\ell+1}} f_{\ell m p}\, h^*_{\ell 0 p},$$

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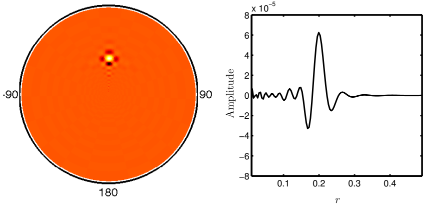
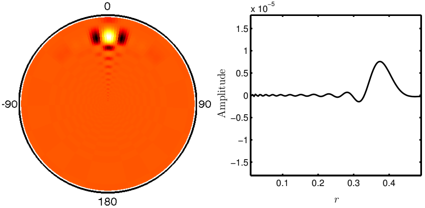
Fourier-Laguerre translation and convolution

- Angular (radial) aperture of localised functions is invariant under radial (angular) translation.



(a) Wavelet kernel translated by $r = 0.2$



(b) Wavelet kernel translated by $r = 0.4$

Figure: Slices of an axisymmetric flaglet wavelet kernel plotted on the ball of radius $R = 0.5$.

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Fourier-LAGuerre wavelets (flaglets) on the ball

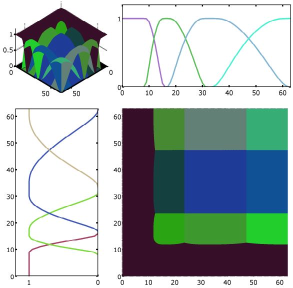

Figure: Tiling of Fourier-Laguerre space.

- *Exact wavelets on the ball* (Leistedt & McEwen 2012).
- Extend the idea of scale-discretised wavelets on the sphere (Wiaux, McEwen, Vandergheynst, Blanc 2008) to the ball.
- Construct wavelets by tiling the ℓ–p harmonic plane.
- Scale-discretised wavelet $\Psi^{jj'} \in L^2(B^3)$ is defined in harmonic space:

$$\Psi^{jj'}_{\ell m p} \equiv \sqrt{\frac{2\ell+1}{4\pi}}\, \kappa_\lambda(\ell\lambda^{-j})\kappa_\nu(p\nu^{-j'})\delta_{m0}.$$

- Construct wavelets to satisfy a resolution of the identity:

$$\frac{4\pi}{2\ell+1}\left(|\Phi_{\ell 0 p}|^2 + \sum_{j=J_0}^{J} \sum_{j'=J'_0}^{J'} |\Psi^{jj'}_{\ell 0 p}|^2 \right) = 1, \ \forall \ell, p.$$

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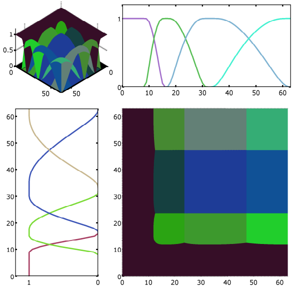

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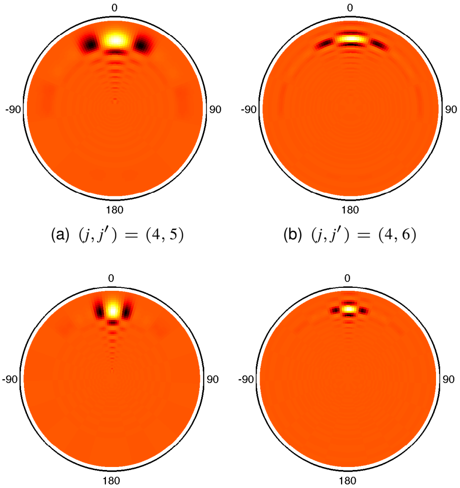


(a) $(j, j') = (4, 5)$ (b) $(j, j') = (4, 6)$

(c) $(j, j') = (5, 5)$ (d) $(j, j') = (5, 6)$

Figure: Scale-discretised wavelets on the ball.

Fourier-LAGuerre wavelets (flaglets) on the ball

Wavelets

Fourier-LAGuerre wavelets (flaglets) on the ball

- The Fourier-Laguerre wavelet transform is given by the usual projection onto each wavelet:

$$W^{\Psi^{jj'}}(\boldsymbol{r}) \equiv (f \star \Psi^{jj'})(\boldsymbol{r}) = \langle f | \mathcal{T}_{\boldsymbol{r}} \Psi^{jj'} \rangle_{\mathbb{B}^3} = \int_{B^3} \mathrm{d}^3\boldsymbol{r}' f(\boldsymbol{r}') (\mathcal{T}_{\boldsymbol{r}} \Psi^{jj'})(\boldsymbol{r}') \, .$$

- The original function may be synthesised exactly in practice from its wavelet (and scaling) coefficients:

$$f(\boldsymbol{r}) = \int_{B^3} \mathrm{d}^3\boldsymbol{r}' W^{\Phi}(\boldsymbol{r}') (\mathcal{T}_{\boldsymbol{r}} \Phi)(\boldsymbol{r}') + \sum_{j=J_0}^{J} \sum_{j'=J'_0}^{J'} \int_{B^3} \mathrm{d}^3\boldsymbol{r}' W^{\Psi^{jj'}}(\boldsymbol{r}') (\mathcal{T}_{\boldsymbol{r}} \Psi^{jj'})(\boldsymbol{r}') \, .$$

Accuracy of the flaglet transform

- For a band-limited signal, we can compute Fourier-Laguerre wavelet transforms exactly.

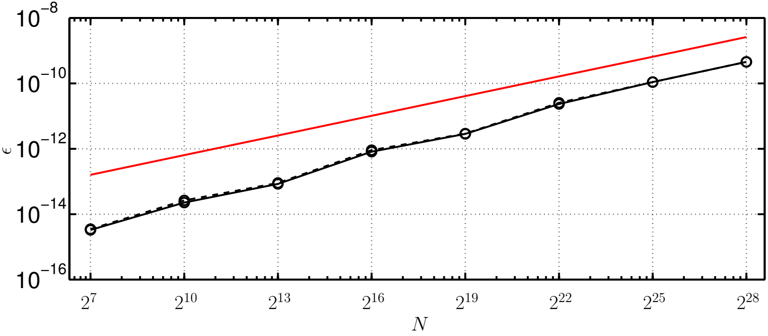


Figure: Numerical accuracy of the flaglet transform.

Computation time of the flaglet transform

- Fast algorithms to compute Fourier-Laguerre wavelet transforms.

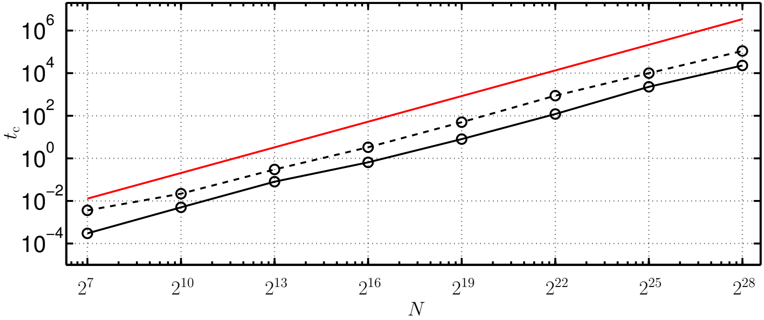
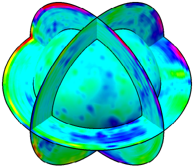


Figure: Computation time of the flaglet transform.

Codes for Fourier-LAGuerre wavelets (flaglets) on the ball

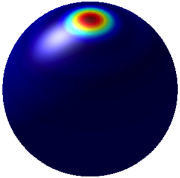


FLAGLET code

`http://www.flaglets.org`

Exact wavelets on the ball
Leistedt & McEwen (2012)

- C, Matlab, IDL, Java
- Exact (Fourier-LAGuerre) wavelets on the ball – coined *flaglets*!



S2LET code

`http://www.s2let.org`

S2LET: A code to perform fast wavelet analysis on the sphere
Leistedt, McEwen, Vandergheynst, Wiaux (2012)

- C, Matlab, IDL, Java
- Support only axisymmetric wavelets at present
- Future extensions:
  - Directional, steerable wavelets
  - Faster algorithms to perform wavelet transforms
  - Spin wavelets

Outline

1 Fourier-Laguerre transform

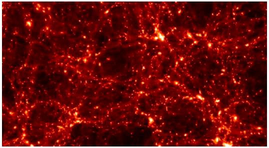
2 Fourier-Laguerre convolution

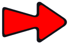
3 Fourier-Laguerre wavelets

4 Illustration

Large-scale structure (LSS) of the Universe

- Map Horizon simulation of large-scale structure (LSS) to Fourier-Laguerre sampling.



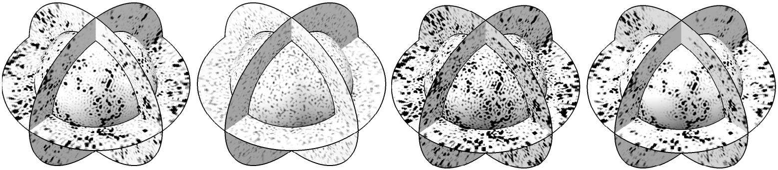


LSS fly through

Flaglet coefficients of large-scale structure (LSS) of the Universe

LSS wavelet coefficients

Flaglet denoising on the ball



(a) Ground truth (b) Noise (c) Noisy (SNR=5dB) (d) Denoised (SNR=11dB)

Figure: Denoising Horizon simulation of large-scale structure (LSS) of the Universe.

Flaglet void finding

- Find voids in the large-scale structure (LSS) of the Universe.
- Perform Alcock & Paczynski (1979) test: study void shapes to constrain the nature of dark energy (*e.g.* Sutter *et al.* 2012).

LSS voids

Summary

- Fourier-Laguerre transform, convolution and wavelets on the ball.
- Fast and exact algorithms.
- All codes publicly available (support C and Matlab):
  - SSHT: `http://www.ssht.org.uk`
  - FLAG: `http://www.flaglets.org`
  - S2LET: `http://www.s2let.org`
  - FLAGLET: `http://www.flaglets.org`
- Application to cosmological observations of the large-scale structure (LSS) of the Universe to learn about dark energy. . .